Multivariate Functions (Cont’d.)

- A final case

\[ F(x) = \alpha \left[ \sum_{i=1}^{n} \beta_i x_i^{-\rho} \right]^{-\frac{1}{\rho}} = \alpha \left[ \beta_1 x_1^{-\rho} + \beta_2 x_2^{-\rho} + \ldots + \beta_n x_n^{-\rho} \right]^{-\frac{1}{\rho}} \]

- and \( \sigma < 0 \)

- Used to describe production possibilities

- Called the Constant Elasticity of Transformation function, or CET

<table>
<thead>
<tr>
<th>CES is increasing in each of its arguments and concave (strictly only in the case of decreasing returns to scale)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CET is also increasing in each of its arguments, but is convex</td>
</tr>
<tr>
<td>If ( \alpha=1, \beta=1, ) and ( \rho=-2 ) (a CET case), is this function familiar?</td>
</tr>
</tbody>
</table>

\[ 2^2 \quad 2^{-1/2} \]
Multivariate Functions (Cont’d.)

- Multivariate linear response and plateau

\[ F(x) = \min [\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n, \beta] \]

where the \( \alpha_i \) and \( \beta \) are constants.

Multivariate Functions (Cont’d.)

- Like the Leontief, this function can be simulated by the objective of a linear program:

\[ F(x) = \text{maximize } y \]

subject to: \( y \leq \alpha_i x_i, \quad i = 1, 2, \ldots, n \)
Composite Functions and Nesting

- Notice the relationship between the univariate linear response and plateau, the multivariate Leontief, and the multivariate linear response and plateau. Can we write these in the form $F(x) = G[H(x)]$? (Identify $F$, $G$, and $H$.)

- The general name for this type of operation is called the composition of functions, and the result is called a “composite function.”

Composite Functions (Cont’d.)

- Composition is how we get complicated functional forms from a small number of simple functions.

- Consider the flexible functional forms called “generalized quadratics”:
  - Translog
  - Generalized Leontief
  - Fourier
Composite Functions (Cont’d.)

- These are simply composites of the multivariate quadratic with other functions – e.g. for the translog,

\[ F(x) = \alpha_o + \sum_{i=1}^{n} \alpha_i \ln(x_i) + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \beta_{ij} \ln(x_i) \ln(x_j) \]

- This could be written \( F(x)=Q[\ln(x_1),\ln(x_2),\ldots,\ln(x_n)] \) where \( Q() \) is the multivariate quadratic.

---

Composite Functions (Cont’d.)

- Other instances of composite functions include:
  
  - CES (three-fold: constant elasticity, multivariate linear, and constant elasticity again)
  
  - Stone-Geary or linear expenditure system (affine and Cobb-Douglas)
  
    \[ F(x) = \alpha \prod_{i=1}^{n} (x_i - \gamma_i)^{\beta_i} \]

  - Nested CES (see below)
Nested Functions

- A particular type of composite function is called a “nested functional form”

- Nesting is typically used to define production relationships via production functions or preferences via utility functions

- With nesting, the arguments of a production function may be production functions themselves

Nested Functions (Cont’d.)

- Nesting can (and often does) go on for several levels

- Consider a two level example (due to Manne) in the case of a production function that produces “GDP” from capital (K), labor (L), non-electric energy (N) and electric energy (E)
Nested Functions (Cont’d.)

\[ F(K,L,E,N) = \alpha \left[ \beta_1 \left( K^{\delta} L^{1-\delta} \right)^{-\rho} + \beta_2 \left( E^{\gamma} N^{1-\gamma} \right)^{-\rho} \right]^{1/\rho} \]

- Notice that \( K \) and \( L \) are combined using a Cobb-Douglas relationship, and \( E \) and \( N \) are also combined using a different Cobb-Douglas

- The \( K&L \) and \( E&N \) aggregates are then combined using a CES relationship

Nested Functions (Cont’d.)

- CES functions are commonly used functions for nesting

- Note that elasticities of substitution across nests are generally not constant even when the nesting functions are CES
Convexity of Composite Functions

- Convexity is an important determinant of the behavior of the optimization problems lying behind economic theory.

- Is convexity inherited by composite functions?
  - I.e., if $f(y)$ is convex and $g(x)$ is convex, is $f[g(x)]$ convex?

Convexity of Composite Functions (Cont’d.)

- Let $f(y) = \exp(-y)$ and $g(x) = x^2$. Is the composite function $f[g(x)]$ convex?
  - $f[g(x)] = \exp(-x^2)$.
  - Is this a familiar function?
  - What does its graph look like?
  - Convex?
Convexity of Composite Functions (Cont’d.)

- **Sufficient conditions to guarantee that a composite function will be convex:**
  
  - If \( f(y) \) and \( g_i(x) \) are convex functions, and if \( f(y) \) is increasing in each of its arguments, then the composite function \( f[g_1(x), g_2(x), \ldots, g_m(x)] \) is convex.

Convexity of Composite Functions (Cont’d.)

- **Sufficient conditions to guarantee that a composite function will be concave:**
  
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Recall that the two input CES is concave

Can we use this fact and the results above to show that the \( n \) input CES is concave?

Can we use these results to show that the nested CES is concave?

An important and growing approach to describing relationships is based on implicit functions

The relationship between a set of variables (e.g., \( x \) and \( y \)) cannot be written explicitly (e.g., as \( y = f(x) \)), but must be written implicitly (e.g., as \( g(x,y) = 0 \))

E.g., \( x^2 + y^2 = z \) cannot be written in the form \( y = f(x,z) \) or \( z = f(x,y) \)
Convex Programming

“Convex programming” refers to a special class of problems that are particularly nice from both modeling and computational perspectives.

Important:

- Convex programming problems have exactly one optimal objective value.
- Sometimes convex programming problems have exactly one optimal solution.

Convex Programming (Cont’d.)

Uniqueness of the optimal objective value is important because:

- From a modeling perspective, it means that changes in the objective value for a convex programming problem can be attributed to changes in the model.
- From a computational perspective, if we find a locally optimal solution, we have found the globally optimal value of the objective.
Convex Programming (Cont’d.)

- Convex programming problems have these properties:
  - The feasible region is a convex set, and
  - If the goal is minimization (maximization), the objective function is convex (concave) over the feasible region.

Convex Programming (Cont’d.)

- Consider the problem:

\[
\begin{align*}
\text{minimize} & \quad F(x) \\
\text{subject to} : & \quad g_i(x) \geq 0 \quad i = 1, \ldots, t, \text{ and}
\end{align*}
\]

where \( x \) is an \( n \)-dimensional vector of problem variables.
The following conditions are sufficient to guarantee that the feasible region is convex:

- \( g_j(x) \) is concave for \( j=1,2,\ldots,t \), and
- \( g_j(x) \) is linear for \( j = t+1, t+2, \ldots, m \).

Again, these conditions are sufficient, but not necessary. For instance, the feasible region is still convex if:

- \( g(x) \) may be concave for \( j=1,2,\ldots,t \) only over the region where the other constraints are satisfied, or
- \( g(x) \) may be “equivalent” to linear functions for \( j = t+1, t+2, \ldots, m \).
Convex Programming (Cont’d.)

- A related type of problem can be defined — the *strictly* convex program

- A convex program with a *strictly* convex (concave) objective to be minimized (maximized) is called a *strictly convex program*

Convex Programming (Cont’d.)

- A strictly convex program has

  - Unique objective function value, and
  - Unique optimal variable values $x$
  - I.e., *any* optimal solution is *the* unique solution
  - Added restriction involves only the objective
  - Objective must be strictly convex with respect to *all* problem variables
Lagrange Multipliers: Interpretation and Signs

- Universal interpretation of Lagrange multipliers is “marginal units of the objective per marginal unit of the right-hand side of the constraint”

- Ex. #1 if the objective is in dollars, and the constraint states that acres of land used cannot exceed the acres available,

- Then the Lagrange multiplier on the constraint is in units of dollars per acre of land available

Lagrange Multipliers (Cont’d.)

- Ex. #2 if the objective is in florins, and the constraint states that the cases of tomatoes at warehouse #1 next week shipped out cannot exceed the cases of tomatoes shipped in to warehouse #1 cumulatively,

- Then the Lagrange multiplier is in units of florins per case of tomatoes shipped in cumulatively
Lagrange Multipliers (Cont’d.)

- Objective may not be in convenient units (e.g., utility)
  - Unit analysis used to convert Lagrange multipliers to monetary units
  - Ex. regardless of the units of the objective, if I have the multipliers from a constraint on initial wealth in dollars ($X$) and a constraint on land ($Y$),
  - Then I can find the value of land in dollars by taking the ratio $Y/X$ (What are the units?)

Lagrange Multipliers (Cont’d.)

- Care is needed in interpreting the resulting units
  - It is important to select the constraint for the denominator such that it will always be an active constraint (otherwise we would be dividing by zero)
Consider the signs of Lagrange multipliers

- The signs of Lagrange multipliers are critically important because –
  - They define economic value, and
  - They can indicate failure of optimality conditions

What do the signs of Lagrange multipliers for an optimal solution depend upon?

- The direction of optimization (max or min), and
- The nature of the relationship between right- and left-hand side of the constraint
### Lagrange Multipliers (Cont’d.)

- Signs of Lagrange multipliers for an optimal solution:

<table>
<thead>
<tr>
<th></th>
<th>Maximization</th>
<th>Minimization</th>
</tr>
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<tbody>
<tr>
<td>( \leq )</td>
<td>+</td>
<td>–</td>
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<td>( \geq )</td>
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<td>=</td>
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</tbody>
</table>

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### Q: How do you remember this?

- A: The Lagrange multiplier measures the change in the optimal objective value associated with an increase in the right-hand side of the constraint.

- Why not draw a picture?
Lagrange Multipliers (Cont’d.)

- Maximize $f(x)$ subject to: $x \leq 3$

![Graph of a linear function with $x \leq 3$]

Lagrange Multipliers (Cont’d.)

- Minimize $f(x)$ subject to: $x \leq 3$

![Graph of a decreasing function with $x \leq 3$]
Lagrange Multipliers (Cont’d.)

Minimize \( f(x) \) subject to: \( x \geq 1 \)