DSP Example

- Again, we illustrate the formulation process with an example -- a portfolio problem with revisions of the portfolio allocations

- Problem features:
  - The decision maker starts with $1000
  - The planning horizon is one year
  - There are two decision points, now and six months from now

DSP Example (cont’d.)

- There are three investment options under consideration: a stock mutual fund (S), a bond mutual fund (B), and cash (C)

- There are no transactions costs

- The decision maker is an expected utility maximizer with constant relative risk aversion equal to 2

- At the end of the first six months, one of four states of nature occurs
DSP Example (cont’d.)

Returns by State of Nature

<table>
<thead>
<tr>
<th>State</th>
<th>Prob.</th>
<th>S</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20</td>
<td>1.15</td>
<td>1.05</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>0.40</td>
<td>0.95</td>
<td>1.03</td>
<td>1.00</td>
</tr>
<tr>
<td>3</td>
<td>0.30</td>
<td>1.10</td>
<td>1.04</td>
<td>1.00</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>1.04</td>
<td>1.07</td>
<td>1.00</td>
</tr>
</tbody>
</table>

- Note that state 1 is good for stocks, state 2 is bad for stocks, state 3 is intermediate for both stocks and bonds, and state 4 is good for bonds and mediocre for stocks
- Note that cash is a sure thing

DSP Example (cont’d.)

For convenience, these same 6-month returns states are used for the second 6-months, but the probabilities are different and depend on the outcomes in the first 6-months

Conditional Probabilities of 2nd 6M. States

<table>
<thead>
<tr>
<th>State (1st 6M.)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.5</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.4</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.3</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>
DSP Example (cont’d.)

- Can we diagram this problem?
- What are the initial decisions?
- What are the initial constraints?
- What are the first random outcomes to occur?
- What decisions are made in the second stage?
- What are the constraints limit choices in the second stage?

DSP Example (cont’d.)

- What random events occur after the second stage decisions?
- How many states of nature are there for wealth at the end of the year?
- How can we keep track of the fact that second stage decisions are conditional on the outcomes in the first stage?
- How can we keep track of the fact that second stage constraints
  - Are conditional, and
  - Only limit the variables that are likewise conditional?
DSP Example (cont’d.)

Formulation

What kinds of sets might we need?

- **SETS**
  - i States of nature / 1*4 /
  - v Investment vehicles / s,b,c /
  - cash(v) Just cash / c /
  - notc(v) Other than cash / s,b / ;
DSP Example (cont’d.)

What are the initial (unconditional decisions)?

- POSITIVE VARIABLES
  - x(v) Initial investment allocation;

What are the initial constraints?

- EQUATIONS
  - limcap1 Initial capital limitation;
  - limcap1 .. sum(v,x(v)) =l= 1000;

DSP Example (cont’d.)

What are the first random outcomes to occur?

- TABLE r(i,* ) Returns for first 6 months
  - prob s b c
  - 1 .2 1.15 1.05 1
  - 2 .4 0.95 1.03 1
  - 3 .3 1.10 1.04 1
  - 4 .1 1.04 1.07 1;
DSP Example (cont’d.)

What decisions are made in the second stage?

- **POSITIVE VARIABLES**
  - \( s(v,i) \) Sales at mid-year
  - \( b(v,i) \) Purchases at mid-year
  - \( y(i) \) Cash held for second 6 mos.;

What constrains those decisions?

- **EQUATIONS**
  - \( \text{limsell}(i,v) \) Limits on selling at mid-year
  - \( \text{limcap2}(i) \) Mid-year capital limitations

DSP Example (cont’d.)

- \( \text{limsell}(i,notc) .. s(notc,i) =l= x(notc) ; \)

- \( \text{limcap2}(i) .. \sum \text{notc} b(notc,i) + y(i) =l= \)
  - \( \sum \text{cash} x(cash) + \sum \text{notc} r(i,notc)s(notc,i) ; \)

What random events occur after the second stage decisions?
DSP Example (cont’d.)

TABLE $q(i,j,\ast)$ Conditional returns distribution for second 6 months

<table>
<thead>
<tr>
<th></th>
<th>1.prob</th>
<th>2.prob</th>
<th>3.prob</th>
<th>4.prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.1</td>
<td>.5</td>
<td>.2</td>
<td>.2</td>
</tr>
<tr>
<td>2</td>
<td>.3</td>
<td>.2</td>
<td>.4</td>
<td>.1</td>
</tr>
<tr>
<td>3</td>
<td>.2</td>
<td>.3</td>
<td>.3</td>
<td>.2</td>
</tr>
<tr>
<td>4</td>
<td>.3</td>
<td>.3</td>
<td>.3</td>
<td>.1</td>
</tr>
</tbody>
</table>

$q(i,j,v) = r(j,v)$;

(Note the conditional nature of this distribution)

DSP Example (cont’d.)

- How many states of nature are there at the end of the year? (4x4 = 16)
- How do we keep track of the fact that second stage decisions are conditional on the outcomes in the first stage? (By indexing them by the first stage state)
- How can we keep track of the fact that second stage constraints are conditional?
  - Only limit the variables with the same conditions?
  - (Again, indexing.)
DSP Example (cont’d.)

For accounting purposes, these variables and constraints are handy:

- **POSITIVE VARIABLES**
  - \( w(i,j) \) Wealth at year’s end
- **VARIABLES**
  - \( eu \) Objective ;

- **EQUATIONS**
  - \( wdef(i,j) \) Wealth definition constraints
  - \( obj \) Expected utility definition constraint ;

Note the use of \( w(i,j) \) so we can be sure wealth stays positive

Note that the special case \( \rho = 1 \) is handled correctly
DSP Example (cont’d.)

- Do the dynamics of these decisions matter?
- Does the conditional nature of response matter?

Experiments

- Compare the results of the DSP to the case where no mid-year adjustment is permitted
- Compare the results of the DSP to the case where mid-year adjustment is allowed but must be the same independent of what happens in the first six months

DSP Example (cont’d.)

- Implementing experiment #1
  
  - \( s.fx(v,i) = 0 \);
  - \( b.fx(v,i) = 0 \);

  - Note that we are not allowing purchases or sales at mid-year
DSP Example (cont’d.)

- Implementing experiment #2
  - equations
    - unifb(v,i) Equalize second period purchases
    - unifs(v,i) Equalize second period sales;
  - unifb(v,i) .. b(v,i) =e= b(v,'1') ;
  - unifs(v,i) .. s(v,i) =e= s(v,'1') ;

- Note purchases and sales must all equal the purchase/sale that occurs after state 1 is realized

DSP Example (cont’d.)

- Results – Initial allocations
  - Stocks     Bonds
    - Expt. #1  902.387  97.613
    - Expt. #2  363.645  636.355
    - DSP       252.216  747.784

- Note that the effect on the initial allocations is quite large – Do you expect conditional transactions to be significant?
DSP Example (cont’d.)

- Results – conditional purchases and sales
  - Expt. #1 – No purchases or sales allowed
  - Expt. #2
  - Bond sales at mid-year 636.355 shares
  - Stock purchases at mid-year 655.445 shares
  - Cash holdings by state of nature 12.727, 0, 6.364, and 25.454 for state 1, 2, 3, and 4

DSP Example (cont’d.)

- DSP conditional purchases and sales
  - State          1  2  3  4
  - Sell Stock     252.2
  - Sell Bond     747.8  630.2  747.8
  - Buy Stock     770.2  655.4  800.1
  - Buy Bond      290.0
DSP Example (cont’d.)

- Expected utility levels
  - Expt. #1: -0.915
  - Expt. #2: -0.913
  - DSP: -0.908

- What do these mean?

DSP Example (cont’d.)

- Alternative concept – certainty equivalent
  - Define: the certainty equivalent is the amount of certain money that makes the decision maker indifferent between the distribution of returns and the certain amount
  
  - Mathematically,

  \[ CE = U^{-1}(E[U(W)]) \]
DSP Example (cont’d.)

- Certainty Equivalents ($):
  - Expt. #1 1092.855
  - Expt. #2 1095.138
  - DSP 1101.565