Nonparametric Efficiency Testing (cont’d.)

More often than not, there will be a mixture of fixed and variable inputs -- then we solve the following:

\[
\begin{align*}
\text{minimize} & \quad \sum_{n=1}^{K} \sum_{n=1}^{K} w_n x_n \\
\text{subject to} & \quad \sum_{k=1}^{K} u_k^k \lambda_k \geq u^0 \\
& \quad \sum_{k=1}^{K} x_n^k \lambda_k \leq x_n \text{ for } 1 \leq n \leq t \\
& \quad \sum_{k=1}^{K} x_n^k \lambda_k \leq x_n \text{ for } n > t, \quad \sum_{k=1}^{K} \lambda_k = 1, \lambda_k \geq 0
\end{align*}
\]

Nonparametric Efficiency Testing (cont’d.)

If we apply this test for firm 1 where input 1 is treated as fixed, we get:

\[
\begin{align*}
\text{minimize} & \quad 2x^2 \\
& \quad 7\lambda_1^1 + 7\lambda_2^2 + 14\lambda_3^3 + 14\lambda_4^4 \geq 7 \\
& \quad 4\lambda_1^1 + 1\lambda_2^2 + 8\lambda_3^3 + 8\lambda_4^4 \leq 4 \\
& \quad 3\lambda_1^1 + 5\lambda_2^2 + 5\lambda_3^3 + 6\lambda_4^4 \leq x^2 \\
& \quad \lambda_1^1, \lambda_2^2, \lambda_3^3, \lambda_4^4 \geq 0
\end{align*}
\]
Nonparametric Efficiency Testing (cont’d.)

- The optimum for this problem has an objective value of 6

- Does firm 1 appear to be minimizing variable costs? (Remember that firm 1 was using 3 units of input 2)

- Recall that when both inputs were variable firm 1 appeared to be inefficient

Nonparametric Efficiency Testing (cont’d.)

- Thus far, we have seen tests for
  - Technical efficiency in the production of one output
  - Allocative efficiency in the minimization of costs

- There are three other types of efficiency tests:
  - Multiple output technical efficiency
  - Profit maximization
  - Revenue maximization, given input levels
Nonparametric Efficiency Testing (cont’d.)

- Multiple output technical efficiency test:

  maximize \( \tau \)  
  \( \tau, \lambda^1, ..., \lambda^K \)  

  subject to:  
  \( \sum_{k=1}^{K} u_m^k \lambda^k \geq u_m^0 \tau \)  
  \( \sum_{k=1}^{K} x_n^k \lambda^k \leq x_n^0 \)  
  \( \sum_{k=1}^{K} \lambda^k = 1, \lambda^k \geq 0 \)

Here, we are asking if the entire vector of outputs can be scaled up by a single factor.

We are not asking whether more of any output could be produced.

What will the optimal objective value be for

- An efficient producer?
- An inefficient producer?
Nonparametric Efficiency Testing (cont’d.)

- Note that we are testing each output at a time for efficiency (0B/0A) in this case.

Here is a case where this test produces an unintuitive result.

---

Nonparametric Efficiency Testing (cont’d.)

- Here is a case where this test produces an unintuitive result.
Nonparametric Efficiency Testing (cont’d.)

For each \( m' \) we could solve the following:

\[
\text{maximize } \tau_{m'} \\quad \tau_{m'}, \lambda^1, \ldots, \lambda^K
\]

subject to:

\[
\sum_{k=1}^{K} u_m k \lambda^k \geq u_m^0 \tau_{m'}
\]

\[
\sum_{k=1}^{K} u_m k \lambda^k \geq u_m^0
\]

\[
\sum_{k=1}^{K} x_n k \lambda^k \leq x_n^0, \quad \sum_{k=1}^{K} \lambda^k = 1, \quad \lambda^k \geq 0
\]
Nonparametric Efficiency Testing (cont’d.)

Now consider our unintuitive example

Profit maximization -- looks much like cost min test

\[
\begin{align*}
\text{maximize} & \quad \sum_{m=0}^{M} p_m u_m - \sum_{n=0}^{N} w_n x_n \\
\text{subject to} & \quad \sum_{k=1}^{K} u^m_k \lambda^k \geq u^m \\
& \quad x^n_k \lambda^k \leq x^n, \quad \lambda^k = 1, \quad \lambda^k \geq 0
\end{align*}
\]
Nonparametric Efficiency Testing (cont’d.)

- Notice that this is the multiple output case
- Notice the similarity with the cost min test
- How would you identify inefficiency?
- Where is the data in the problem that is relevant to the individual firm?
- How does this change with fixed inputs?

Nonparametric Efficiency Testing (cont’d.)

- If we apply this test for firm 2 where the output price:

\[
\begin{align*}
\text{minimize} & \quad 4u - 4x_1 - 3x_2 \\
7\lambda_1^1 + 7\lambda_2^2 + 14\lambda_3^3 + 14\lambda_4^4 & \geq u \\
4\lambda_1^1 + 1\lambda_2^2 + 8\lambda_3^3 + 8\lambda_4^4 & \leq x_1 \\
3\lambda_1^1 + 5\lambda_2^2 + 5\lambda_3^3 + 6\lambda_4^4 & \leq x_2 \\
\lambda_1^1, \lambda_2^2, \lambda_3^3, \lambda_4^4 & \geq 0
\end{align*}
\]
Nonparametric Efficiency Testing (cont’d.)

The optimal solution for this problem has an objective value of 9

Is firm 2 efficient?

Yes – observed profit was 9

How would this change if input 1 was fixed?

Test of profit max with input 1 fixed – Firm 2

minimize $4u - 2x^2$

\[ 7\lambda_1^1 + 7\lambda_2^2 + 14\lambda_3^3 + 14\lambda_4^4 \geq u \]

\[ 4\lambda_1^1 + 1\lambda_2^2 + 8\lambda_3^3 + 8\lambda_4^4 \leq 1 \]

\[ 3\lambda_1^1 + 5\lambda_2^2 + 5\lambda_3^3 + 6\lambda_4^4 \leq x^2 \]

New optimal objective is 18, and firm 2 is efficient
Nonparametric Efficiency Testing (cont’d.)

- Revenue maximization test

\[
\text{maximize } \sum_{m=1}^{M} p_m^0 u_m \\
\text{subject to: } \sum_{k=1}^{K} u_m^k \lambda^k \geq u_m \\
\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n^0, \quad \sum_{k=1}^{K} \lambda^k = 1, \quad \lambda^k \geq 0
\]

Nonparametric Efficiency Testing (cont’d.)

- Again, this is similar to cost min/profit max problems and looks as follows for firm 2:

\[
\text{minimize } 4u \\
7\lambda^1 + 7\lambda^2 + 14\lambda^3 + 14\lambda^4 \geq u \\
4\lambda^1 + 1\lambda^2 + 8\lambda^3 + 8\lambda^4 \leq 1 \\
3\lambda^1 + 5\lambda^2 + 5\lambda^3 + 6\lambda^4 \leq 5
\]
Nonparametric Efficiency Testing (cont’d.)

- In the single output case, is the revenue test strange?

- Note that technical inefficiency implies profit and revenue maximization inefficiency

- Cost minimization inefficiency implies profit maximization inefficiency

- Revenue maximization inefficiency implies profit maximization inefficiency

Nonparametric Efficiency Testing (cont’d.)

- How do we add the assumption of constant returns to scale?

- There is a problem with the profit maximization test when all inputs are variable and we assume constant returns to scale. What is it?
Nonparametric Efficiency Testing (cont’d.)

- Want to learn more? Then read:

  
  

Producer Problems -- Complications

- Incorporating additional activities/constraints in producer problems -- a question of scope

- What makes producer problems interesting is their interactions with the world

  - Market interactions -- purchases and sales of inputs or outputs

  - Resource transactions -- factor rental, service hiring, borrowing
Producer Problems -- Complications

- Regulations
  - Prohibitions or limits on use of some inputs
  - Prohibitions or limits on the production of some outputs

- Contracts with suppliers of input, or demanders of outputs

- Formulation of constraints and objective must reflect
  - Producer incentives
  - Tradeoffs available to producers

Endogenous Price Models

- In some cases, price taking behavior is not the appropriate perspective, e.g.,
  - Where producers have market power
  - Models of entire sectors wherein we know there is a relationship between supply and demand quantities and market prices
Endogenous Price Models (cont’d.)

The key is to know what the relationship between price and quantity is

- Demand, \( Q(P) \), is typically known, but
- What we need is inverse demand, \( P(Q) \); so, the relationship must be invertible

\[ Q(P) = \beta P^\varepsilon \quad \text{and} \quad P(Q) = (Q/\alpha)^{1/\varepsilon} \]

Endogenous Price Models (cont’d.)

In optimization modeling, endogenous price models are often used to model partial equilibrium problems

Simple example:

- Let’s say we know that demand is \( D(P) = aP^b \)
- and supply is \( S(P) = cP^d \)
Now consider the problem:

\[
\text{maximize } \frac{b}{1+b} D^{(1+b)/b} a^{-1/b} - \frac{d}{1+d} S^{(1+d)/d} c^{-1/d}
\]

subject to: \( D - S \leq 0 \)

The first-order conditions for this problem are:

- \((D/a)^{1/b} - P = 0\) (where \(P\) is the shadow price on the constraint),
- \((S/c)^{1/d} - P = 0\), and
- \(D - S \leq 0\)

Notice that we can use algebra to re-express the first order condition for \(D\) as \(D = aP^b\)

And likewise for \(S\), \(S = cP^d\)

So, the shadow price on the constraint is acting like the market price and the relationships between Supply, Demand, and Market Price have been modeled.
Endogenous Price Models (cont’d.)

How did we do this?

$$\text{maximize } \int_{s,D}^{d} (d/a)^{1/k} dd - \int_{s,D}^{s} (s/c)^{1/d} ds$$

subject to: $D - S \leq 0$

So, our objective is the “integral under the inverse demand curve” less the “integral under the inverse supply curve” – producer’s plus consumer’s surplus

Endogenous Price Models (cont’d.)

Graphically, here is what is going on
Endogenous Price Models (cont’d.)

- It is straightforward to adapt this model to analysis of spatial equilibrium problems.
- The base is a transportation problem, transhipment problem, or transhipment problem with processing.
- Demand and (or) supply are no longer treated as fixed, but rather integrals of the inverse relationships are incorporated in the objective.

Endogenous Price Models (cont’d.)

- For example, the transportation model with variable supplies and demands becomes:

\[
\max_{s,t,d} \sum_{j=1}^{m} D_j P_j(d_j)dd_j - \sum_{i=1}^{n} S_i P_i(s_i)ds_i \\
- \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}
\]

subject to: \(x_{ij} \leq D_j, S_i - x_{ij} \leq 0, x_{ij} \geq 0\).
Endogenous Price Models (cont’d.)

- This formulation has the same interpretation as before -- producers’ plus consumers’ surplus

- There is an adjustment for inter-industry costs

- Graphical illustration is no longer feasible

- This type of model is frequently used to study markets for a single good or a sector (partial equilibrium)