

VALUING STRATEGIC GROWTH OPTIONS

A Portfolio Approach

ROBERTO S. VASSOLO, FILIPE DE ALMEIDA RAVARA, AND JOHN M. CONNOR

ABSTRACT: This study analyzes the trade-off between strategic flexibility and commitment for cases of simultaneous and related strategic investments under high levels of uncertainty. It develops a model that, using a Cournot game and real option theory, demonstrates that (1) a correlated strategic investment adds value to a portfolio of ongoing strategic investments in a decreasing marginal fashion, and (2) the new investment delays the development of the other investments. Managers who fail to recognize these properties may make strategic commitments that destroy value, even in the presence of options with individual positive values. An important feature of the model is that competitive advantages may flow from market power or from the capability of managing the portfolio.

RESUMEN: Este estudio analiza el trade-off entre flexibilidad estratégica y compromiso para el caso de inversiones estratégicas simultáneas e interrelacionadas bajo altos niveles de incertidumbre. Presenta un modelo que, utilizando un juego de Cournot y teoría de las opciones reales, demuestra que (1) una inversión estratégica correlacionada agrega valor al portafolio de las actuales inversiones estratégicas de manera decreciente; y (2) la nueva inversión retarda el desarrollo de las demás inversiones. Los ejecutivos que fallan en reconocer estas propiedades podrían realizar inversiones estratégicas que destruyan valor, aún cuando estas tengan un valor de opción individual positivo. Una característica importante del modelo es que la ventaja competitiva puede emerger del poder de mercado o de la capacidad de manejar el portafolio.

The real options approach to strategic decisions has become a powerful tool in analyzing strategic commitments under uncertainty. Most studies that follow this approach within the strategic management field share a salient characteristic—the independence of strategic investments with other ongoing strategic investments (Chang, 1995; Folta, 1998; Kogut,

1991; Mang, 1998). That is, they do not allow for the existence of interactions within a portfolio of strategic investments. However, it is widely recognized that firms typically consider a set of simultaneous strategic investments in similar strategic domains, and that these investments often exhibit important correlations (Madhok, 1997). Several different studies show that extending single option analysis to multiple options analysis, in the presence of correlations between the underlying assets, is far from straightforward (Johnson, 1987; McGrath, 1997; Stulz, 1982; Trigeorgis, 1996). Multiple options analysis is even more challenging when the firm faces rivalry for the investment opportunity (Kulatilaka & Perotti, 1998).

The purpose of this paper is to investigate the trade-off between flexibility and commitment in the case of multiple and simultaneous investments in oligopolistic markets. By combining concepts from oligopoly and real options theory, this paper builds a model of simultaneous compound options that mimics the strategic process of firms' investments across alternative market environments. The solution of the model is a decision rule that optimizes the choice of a portfolio of strategic investments.

The model loosens three common assumptions found in previous real options studies: (1) the firm has monopoly over an investment opportunity, (2) the product market is perfectly competitive, and (3) the value of one strategic option is independent of the value of other simultaneous strategic investments. Kulatilaka and Perotti (1998) relax the first two

Roberto S. Vassolo is an assistant professor of Business Policy and Strategy at IAE, Universidad Austral. He received his Ph.D. in Strategic Management at Purdue University. He specializes in strategies under uncertainty, strategic alliances, real option thinking, and dynamic capabilities. In particular, he studies the strategic problems faced by high-tech industries that compete in emerging economies.

Filipe de Almeida Ravara is a general manager and assistant, Board of the Assessment Management Division, Caixa Geral de Depósitos (CGA), and head of communications, strategic development, and e-business at CGA. He received a Ph.D. in Agriculture Economics and an M.S. in Horticulture Economics at Purdue University. He specializes in strategic management and marketing economics/price analysis.

John M. Connor is a professor at Purdue University. He received a B.S. in Mathematics from Boston College, an M.S. in Agricultural Economics at the University of Florida, and a Ph.D. in Agricultural Economics at the University of Wisconsin. Dr. Connor specializes in empirical research on market competitiveness and antitrust policy.

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assumptions within a compound option model. This current study generalizes their work by relaxing the independence assumption.

Following Kulatilaka and Perotti (1998), this paper assumes that an initial investment results in the acquisition of a capability that allows the firm take better advantage of future growth opportunities. The source of option value comes from the cumulative nature of this resource or capability. Given that the capability is path-dependent, it is very difficult or costly to make instantaneous adjustments of its level (Dierickx & Cool, 1989). A firm can internalize the benefits of better market conditions if and only if it has already developed certain resources or capabilities—that is, has bought an option. As the main purpose of this study is to analyze the effect of correlation between strategic options, the option is analyzed in relationship with the rest of the portfolio. The critical feature of the portfolio is that the options are seeking a first-mover-type of competitive advantage. Framed in this way, the exercise of one option can negatively affect the value of the remaining options of the portfolio. In addition, it affects the value of the firm competitors' similar options. This is the case, for example, of a pharmaceutical company carrying out research in a specific biotechnological domain through different strategic alliances. Discovery of a drug by one alliance diminishes or eliminates the value of the remaining related alliances and that of competitors' ongoing research in a similar technological domain.

The model's two main findings are that the value of a new strategic option in a correlated portfolio of investments in an oligopolistic market has decreasing marginal returns, and that the exercise of one option delays the exercise of other options. A commendable feature of the model is that previous real options models can be shown to be special cases. More important, the present model highlights the possibility of having strategic investments that create value in isolation, but destroy value when they are considered within a portfolio of related investments. In addition, the model generates market power and resource-based types of competitive advantages, characteristics not fully explored in previous models.

A MODEL OF SIMULTANEOUS GROWTH OPTIONS

The task of building competitive capabilities and strong market positions requires significant resource commitments. Typically, a single firm cannot amass sufficient resources to finance expansion in all profitable market opportunities. Partial commitments often enable the firm to create an upper bound on the risks associated with placing the bet so that the opportunity will be realized, often by decreasing the total investment cost to an affordable level. If the decision to make partial commitments derives from the prospect of expanding into a new

product or market, which can materialize after additional information reduces environmental uncertainty, then this decision is equivalent to buying a call option. As stated above, this study is concerned with those types of investments that provide to the firm a first-mover competitive advantage (e.g., obtaining a patent). Therefore, when the firm has a portfolio of this type of simultaneous investments, the decision to exercise the option is likely to generate a loss in the value in all the remaining correlated options. Similarly, in the presence of competitors with similar strategic approaches and in related products and markets, the decision to exercise this option is likely to generate a loss on the value in all the remaining competitors' equivalent options and may induce competitors' divestiture of particular options. Divestiture arises because the value of the investment includes both the cash flows stemming from current assets and those stemming from asset redeployment or future expansion (Myers, 1977). The latter cash flows are only realized if the business is expanded; therefore, exercising the option requires further commitment.

The model distinguishes four investment alternatives: full strategic commitment, minor resource commitment ("buying" a real option), major resource commitments ("exercising" the option), and divestitures (abandoning the unexercised option). This paper assumes that two critical differences distinguish buying an option (minor resource commitment) from fully committing resources. First, when the firm buys an option, investments are not entirely irreversible. Second, only full commitments generate current income, which can be appropriated by the firm. That is, the option value is contingent upon the value of the corresponding major resource commitment. Real options are temporary investment vehicles, which will either disappear or be converted into a full strategic commitment at some point in time. It is worth noting that the only difference between a full strategic commitment and a major resource commitment is the investment sequence. Whereas the former consist in a "one-shot" investment, the latter is conditional to the previous existence of a minor resource commitment. After the major resource commitment is done, it generates the same cash flow stream as a similar ongoing full resource commitment. In fact, every major or full resource commitment is an option on future market growth (a compound option).

The model departs from previous ones in several aspects. Most previous real option models assume that each investor owned a single undeveloped asset, which could become a developed asset by paying a strike price or development cost (Brennan and Schwartz, 1985; Paddock, Siegel, Smith, & Zeithaml, 1988; Titman, 1985). Departures from this assumption have been scarce (e.g., Dixit & Pindyck, 1994; Trigeorgis, 1996; Williams, 1992) yet are critical if one is to incorporate the effects of strategic interaction among firms into the analysis of their investment decisions. The model also departs from

the proprietary/single asset perspective. First, it assumes that firms compete to accumulate capacity and fulfill demand within each market. Second, it assumes that firms may have simultaneous access to multiple real options within a market. Ultimately, they may decide to develop only a subset of these options into a major resource commitment.

In the model, each firm has initial access to a portfolio of real options within the market for those options' output. Market size (demand) is assumed to grow stochastically over time. At some point in time, the firm may decide to exercise a subset of its options within the market, converting them into major resource commitments. The firm will then use these investments as vehicles to expand its activities within the market. In subsequent periods, the firm will further develop internally these investments, trying to catch up with market growth. The remaining options will lose strategic importance to the firm and will be divested. As long as the market keeps growing over time, the firm can always marginally increase its market presence. Thus, the option to expand in the market never expires, and each new round of financing corresponds to exercising an additional stage in a compound option with infinite stages.

Each firm is a player in a Cournot game with conjectural variations specific to each of its options' output markets. A firm's current market sales are derived from an expression for the optimal output from aggregate investment sales in that market. In order to attain its optimum output goal in a market, the firm develops an optimal number of fully committed strategic investments. The optimum is derived as the Cournot solution to a capital accumulation game involving all the firms in the market.

Three stylized facts of firms' growth are explicitly incorporated in the options model. (1) When a firm behaves as an oligopolist, its actions can affect product market prices, and thereby the sales of all firms in the market. How each firm responds in equilibrium depends partly on the degree of rivalry among firms. (2) If firms face increasing marginal costs of option development, then the aggregate cost of option development depends on the aggregate demand for option development, which is a function of the untapped demand for the aggregate output from developed options. The resulting equilibrium affects each firm's optimum exercising policy and imposes restrictions on the number of real options that are actually developed into major resource commitments. (3) The model permits firms to hold a portfolio of real options within each market. A major resource commitment causes the upside potential value of the remaining options belonging to the firm's portfolio to fall. Therefore, the calculation of the net capital gain from developing options takes into account the expected negative effect of development upon the value of the firm's undeveloped option portfolio, as well as the effect of possible retaliation by the firm's competitors with similar

options. This stylized feature is critical, as a failure to consider the expected negative effect on the firm's undeveloped option portfolio would induce investments that destroy value.

The model is set up in three stages. First, it derives from Cournot first-order profit maximization conditions an expression for each project's current income. Second, it addresses the issue of change in a project's income over time. This implies determining expressions for market growth and for the dynamics of full investments supply. Third, the results from the first two stages are incorporated into two option valuation expressions to determine the market values of the portfolios of options and full commitment investments.

Project Current Income

Consider a market segment s within which the firm i develops K_{si} majority investments during period t . The set of these K_{si} investments is designated as a full commitment investment si . The profit function of a full commitment investment si within market segment s , and the corresponding Cournot first-order profit maximization conditions, are given by the following expressions:

$$\Pi_{si} = P_s q_{si} - TVC(q_{si}) - c_{si} \quad (1)$$

$$\begin{aligned} d\Pi_{si} / dq_{si} &= P_s + q_{si} \left[(dP_s / dQ_s) (1 + dQ_{sj} / dq_{si}) \right] \\ &\quad - MC(q_{si}) = 0, \end{aligned} \quad (2)$$

where Π_{si} is the profit of full commitment investment si within s , P_s is the market price of output from full commitment investment si within market segment s , q_{si} is the output from full commitment investment si within s , q_{sj} is the output from other full commitment investment within s ($j \neq i$), $Q_s = \sum_{j \neq i} q_{sj} + q_{si}$ is the aggregate output from all firms' full commitment investments within market segment s , $TVC(q_{si})$ is firm i 's total variable cost function within s , c_{si} is firm i 's total fixed cost of developing a full commitment investment within s , $MC(q_{si})$ is firm i 's marginal cost function within s , and $dQ_{sj} / dq_{si} = \sum_{j \neq i} dq_{sj} / dq_{si}$ is the conjectural derivative of q_{sj} with respect to q_{si} .

The expression for full commitment investment si 's marginal revenue from Equation (2) can be transformed as follows (Scherer & Ross, 1990: 229):

$$\begin{aligned} &P_s - q_{si} \left[(dP_s / dQ_s) (1 - dQ_{sj} / dq_{si}) \right] \\ &= P_s - (P_s / e_s) s_{si} (1 - dQ_{sj} / dq_{si}), \end{aligned} \quad (3)$$

where e_s is the own-price output demand elasticity within market segment s , and s_{si} is the market share of full commitment investment si .

With a little algebra and some standard assumptions (see Appendix A) it is possible to arrive at the following expression to y_{si} , which is the current income from a full commitment investment:¹

$$y_{si} = (1/\kappa_{si} \nu_{si}^2) \left[(s_{si}/|e|_s) - (s_{si}^2/|e|_s^2) \right] x_s^2 q_s^{-2}. \quad (4)$$

Change in Project Over Time

The exogenous variable market size x_s changes stochastically through time, and it is assumed to follow a geometric Wiener process

$$dx_s/x_s = \mu_s dt - \sigma_s dz, \quad (5)$$

where μ_s and σ_s are the mean and standard deviation of the growth rate in the exogenous market size x_s , and the variables t and z are time and a standard Wiener process. As it is assumed that the market growth rates dx_s/x_s are correlated across market segments, the values of different categories of projects can be correlated in equilibrium.

At all times, there are n firms operating within each segment s , where a segment is one line of business in one geographic area. These firms are assumed to develop majority investments in accordance with a Cournot capital accumulation game. This is a prototypical model that has an extremely simple structure, but that has basic properties that are the same as those of more general models (Tirole, 1988: 315). Consider a market segment s within which firm i competes with $(n_s - 1)$ other firms. Let the demand for capital $K_s = \beta_s P_s$, where P_s is the market price of capital, and let us define each firm i 's total capital accumulation cost function as

$$TC_{si} = c_{si} K_{si}. \quad (6)$$

K_s is the number of majority investments developed by firm i within s , and c_{si} is the investment marginal cost of firm i . If c_{sj} is set at $c_{sj} = 1$, for all $j \neq i$, then $C_{si} = c_{si}/c_{sj}$ can be interpreted as the relative cost disadvantage of firm i over its competitors. Given Equation (11) and other restrictions to c_{si} (see Appendix B), the optimal levels of K_{si} and $K_{s2} = \sum_{j \neq i} K_{sj}$ are as follows:

$$K_{si} = (1/(n_s - 1)) (\beta_s - 1 - n_s (C_{si} - 1)) \quad (7)$$

$$K_{s2} = (1/(n_s + 1)) ((\beta_s + C_{si})(n_s - 1) + 2). \quad (8)$$

If all firms invest at their optimal Cournot level over period t , then

$$K_{si} - K_{s2} = (n_s/(n_s - 1)) (\beta_s - (C_{si} - n_s - 1)/n_s). \quad (9)$$

Total development cost by firm i is given by $K_{si} C_{si}$. Notice that an increase in the cost disadvantage C_{si} leads to a reduction in the size of the full commitment investment K_{si} developed by firm i . The restriction that total development cost of firm i is increasing in C_{si} , or $dT(C_{si})/dC_{si} > 0$, for all C_{si} , is imposed. This holds for all β_s and C_{si} , such that $C_{si} < (1/2n_s)(\beta_s - 1 - n_s)$.

The aggregate supply q_s of majority investments changes deterministically through time, in accordance with the following expression:

$$dq_s/q_s = b_s = K_{s2} - I_{si} K_{si}, \quad (10)$$

where b_s is the aggregate observed development rate within a specific segment s , and I_{si} is a decision variable for firm i , corresponding to the proportion of K_{si} that it effectively decides to develop during period t , according to the option model's solution.

Market Value of Full Commitments and Real Options

The current market value of each minor resource commitment (real option) and full commitment investment can be specified as $Vu_{si}(q_s, x_s)$ and $Vd_{si}(q_s, x_s)$, respectively. The number of options held by each firm i within market segment s at the end of period t is designated by ω_{si} . The risk-adjusted expected growth rate for y_{si} is defined as $\rho_{si} = \mu_s - \lambda_{si} \sigma_s$, where μ_s and σ_s were defined in Equation (5). The parameter $\lambda_{si} \geq 0$ is a coefficient of risk aversion for firm i , equal to the excess mean rate of return required by investors in each market segment. This parameter is equal to 0 for risk-neutral investors, and positive for risk-averse investors. To ensure that the values of the real options and the full commitment investments are positive and finite, the risk-adjusted mean ρ_{si} must be upper-bounded $\rho_{si} < \iota_{si}$, where ι_{si} is the risk-free interest rate available to firm i . The valuation functions for a full commitment investment and a minor resource commitment (real option investment) are given by the following Black and Scholes expressions (see Appendix C):

$$0 = \frac{1}{2} \sigma_{si}^2 x_s^2 VD_{sxxx} + (K_{s2} + I_{si} K_{si}) q_s Vd_{siq} + \rho_{si} x_s Vd_{sxx} - \iota_{si} Vd_{si} + 1 / [(\kappa \nu^2) (1 + (1/e)_s) s_{si}] x_s^2 q_s^{-2} \quad (11)$$

and

$$0 = \max I_{si} \left\{ \frac{1}{2} \sigma_{si}^2 x_s^2 Vu_{sxxx} + (K_{s2} + I_{si} K_{si}) q_s Vu_{siq} + \rho_{si} x_s Vu_{sxx} - \iota_{si} Vu_{si} + K_{si} \omega^{-1} (Vd_{si} - Vu_{si} - K_{si} C_{si}) \right\}, \quad (12)$$

where the subscripts x and q symbolized derivatives.

In addition, Vd_{si} and Vu_{si} are subject to the following constraints:

$$Vd_{si}(q,0) = Vu_{si}(q,0) = 0 \quad (13)$$

$$Vd_{si}(q,0) \leq Z / (\kappa_{si} \nu_s^2) (1 + (1/e_s) s_{si}) x_s^2 q_s^{-2} \quad \text{as } x \rightarrow \infty \quad (14)$$

and

$$Vu_{si}(q,0) \leq Z / (\kappa_{si} \nu_s^2) [1 + (1/e_s) s_{si}] x_s^2 q_s^{-2} \quad \text{as } x \rightarrow \infty. \quad (15)$$

Constraint (13) is an implication from Equation (5): if x is ever zero, it will remain zero, so the project has no value. Conditions (14) and (15) just say that if full commitment investment and real option are to have a finite price-earnings ratio, then their value per unit of income must be bounded above by some positive constant Z . Together, Equations (11), (13), and (14) determine the value of a full commitment investment, whereas Equations (12), (13), and (15) determine the value of the real option.

Equation (12) differs from the corresponding equations for financial options because, as Vd_{si} depends on the market elasticity e_s and firm's i market share s_{si} , developing an option into a majority investment (or calling and option) affects Vd_{si} . This does not happen in financial options' models, where the options have zero net supply. Another particularity of this real option model, stated in the first section, is that the options to develop real assets can be exercised in Equation (12) no more rapidly than the development rate of each developer. This illustrates the path-dependent nature of the resources and capabilities.

The valuation functions of Equations (11) and (12) can be solved for $Vd_{si}(q,x)$, $Vu_{si}(q,x)$, and a switching point y_{si} , which represents the current income level at which firm i develops an option into a major commitment. At the solution to Equation (12), each firm i develops minority (options) into major commitments at rate $I_{si}^* K_{si}$. The last term in Equation (11) is the current income y_{si} from a full commitment investment. To understand the last term in Equation (12), note that firm i converts $I_{si}^* K_{si}$ option into major commitment per unit of time, using its portfolio of ω_{si} options within market segment s . Firm i , therefore, converts $I_{si} K_{si} / \omega_{si}$ units of its option portfolio within s into one full commitment investment per unit of time. This generates a capital gain of $Vd_{si} - Vu_{si} - C_{si} K_{si}$ per full commitment investment, which is equivalent to a capital gain of $K_{si} \omega^{-1} (Vd_{si} - Vu_{si} - C_{si} K_{si})$ per option. This gain reflects the dependence of the value of undeveloped options Vu_{si} on the concurrent value of developed options Vd_{si} . With imperfectly competitive firms, this gain may be positive, whereas it should be zero under perfectly competitive condi-

tions. In other words, in this model, market power is one source of competitive advantage.

MODEL SOLUTION

The previous section provides the valuation functions for different types of commitment. The next step is to determine the decision rule for whether to partially or fully commit resources. Following Williams (1992) and Ravara (1994), the model is solved in five steps. First, it is temporarily assumed that there is a finite and nonnegative level of current income $y = y^*$ from each full commitment investment, above which firm i is willing to convert an option j into the corresponding major commitment j . Second, Equations (11), (13), and (14) are solved for F , the asset value of full commitment investment. Third, the expression for F is plugged into Equation (12), and Equations (12), (14), and (15) are solved for G , the value of the options. Fourth, the values of F and G are used to find the switching point y^* . Fifth, the switching point y^* is shown to verify the temporary assumption in the first step above.

In order to solve the investment problem above, the following transformations are made: $Vd(q,x) = F(y)$, and $Vu(q,x) = G(y)$. With these transformations, $F(y)$ and $G(y)$ are the current market values of the full commitment investments and the real option investments, respectively, conditional only on y per full commitment investment. Because both Equations (11) and (12) are linear on the decision variable I_{si} , then the optimal I_{si}^* will be either zero or one (Dixit & Pindyck, 1994). The following inequalities are temporarily assumed to hold:

$$I_{si}^* \begin{cases} 0 & \text{if } 0 \leq y_{si} \leq y_{si}^* \\ 1 & \text{if } y_{si}^* \leq y_{si} \leq \infty \end{cases} \quad (16)$$

In accordance with Equation (16), I_{si} is redefined as $I_{si}(y)$, such that $I_{si}(y) = 0$ if $0 \leq y \leq y^*$, and $I_{si}(y) = 1$ if $y^* \leq y \leq \infty$. With Equation (16) and the transformation suggested above, the valuation functions (11) and (12) are given by expressions (17) and (18):

$$0 = \frac{1}{2} \sigma^2 y^2 F'' + (\rho + K_2 + IK_i) y F' - \iota F + y \quad (17)$$

$$0 = \frac{1}{2} \sigma^2 y^2 G'' + (\rho + K_2 + IK_i) y G' - \iota G + K_i / \omega (F - G - K_i) I, \quad (18)$$

where K_i and K_2 were defined in expressions (7) and (8), respectively. The subscripts were dropped to simplify the exposition. The last term in Equation (18) is the capital gain

resulting from the development of a D -position. This value is zero for $\omega = \infty$, and different for zero otherwise. After rearranging terms, Equation (18) becomes

$$0 = \frac{1}{2}\sigma^2 y^2 G'' + (\rho + K_2 + IK_i) y G' - (\iota + IK_i / \omega) G + (\beta_s + 1 - 2C) / (3\omega) (F - K_i) I. \quad (19)$$

Let π represent a generic parameter, and consider parameter values that satisfy the assumed inequalities in Equation (19). The coefficient of the control $I_{s^i}(y)$ in Equation (19) is proportional to the value $H(y|\pi)$ from the following function:

$$H(y|\pi) = F - G - \omega y G' - K_i. \quad (20)$$

Notice that if Equation (16) is true, then y^* is the current income level at which the firm is indifferent between keeping a U - or a D -position on market segment s over time period t —it is the solution to $H(y^*|\pi) = 0$. Moreover, given Equation (16), this root satisfies the condition $H'(y^*|\pi) > 0$. Under these conditions, it can be concluded that a sufficient condition for development to be delayed when the value of parameter π increases is that $H(y^*|\pi)$ be decreasing in π . The conditions under which $H(y^*|\pi)$ is decreasing on a parameter π can be seen by examining Equation (20). F is the value of a D -position, and G is the value of a U -position. The third term, $\omega y G'$, is the effect of developing a D -position on y , and thereby upon the value of the remaining U -positions in the firm's portfolio. When expressed as a function of q and x , the third term in Equation (20) assumes the following form:

$$(\omega_{s^i} / v_s \kappa_{s^i}^2) \left[(s_{s^i} / |e|_s) - (s_{s^i}^2 / |e|_s^2) \right] x_{s^i}^x q_{s^i}^{-2} G'. \quad (21)$$

Expression (21) is increasing on x , ω , and $s/|e|$ for $s/|e| < 0.5$, and is decreasing in v , κ , and q . With perfectly competitive ventures ($q = \infty$), the term G' becomes equal to zero. Furthermore, if each real option produces a positive amount of output, then an infinite proportion of real options relative to firm's majority commitments ($v = \infty$) also causes the term G' to be equal to zero. Hence, the following conclusions may be made:

1. With financial options, the first term in Equation (20) is independent from all parameters, and the third term is zero. Therefore, a larger value of a parameter π increases the switching point y^* and thereby defers optimal exercise of the option if and only if it also increases the value of the option at y^* , $G(y^*|\pi)$.
2. With real options and either perfectly competitive conditions among firms or an infinite proportion of independent options relative to firm's majority

commitments, larger parameter values increase the switching point y^* if and only if they also increase the difference between the values of the U - and D -positions at y^* , $G(y^*|\pi) - F(y^*|\pi)$.

3. Finally, with real options and either imperfect competition among firms or a nonzero proportion of firm's options relative to independent options, larger values of π defer the option exercise if and only if they decrease the value of the difference $G(y^*|\pi) - F(y^*|\pi) - \omega y G'(y^*)$.

Equations (17) and (19) are second-order first-degree differential equations and can, therefore, be solved analytically for the values of F and G (see Appendices D and E) so:

$$F(y) = \begin{cases} [r_1 - s_2(r_1 - d_{11})] / [r_1 d_{11}(s_1 - s_2)] y^* (y/y^*)^{s_1} \\ + \xi_1 y & \text{if } 0 \leq y \leq y^* \end{cases} \quad (22)$$

$$F(y) = \begin{cases} [r_1 - s_1(r_1 - d_{12})] / [r_1 d_{12}(s_1 - s_2)] y^* (y/y^*)^{s_2} \\ + \xi_2 y & \text{if } y^* \leq y \leq \infty \end{cases} \quad (23)$$

$$G(y) = \begin{cases} [r_2 - \eta_2(r_2 - d_{21})] / [r_2 d_{21}(\eta_1 - \eta_2)] y^* (y/y^*)^{\eta_1} \\ + \theta_1 \Omega & \text{if } 0 \leq y \leq y^* \end{cases} \quad (24)$$

$$G(y) = \begin{cases} [r_2 - \eta_2(r_2 - d_{21})] / [r_2 d_{22}(\eta_1 - \eta_2)] y^* (y/y^*)^{\eta_2} \\ + \theta_2 \Omega & \text{if } y^* \leq y \leq \infty. \end{cases} \quad (25)$$

Since y^* is the unique root to $H(y) = 0$, its value can be calculated through substituting the values for F , G , and G' into Equation (20), where F , G , and G' are obtained from Equations (23) and (25), for the case $y > y^*$. Equation $H(y^*) = 0$ has the following solution:

$$y^* = \frac{CK_i - \theta_2 K_i^2 C}{A1 - A2 + A3}, \quad (26)$$

with

$$A1 = \frac{r_1 - s_1(r_1 - d_{12})}{r_1 d_{12}(s_1 - s_2)} \left(1 - \theta_2 K_i s_2 - \theta_2 \frac{K_i}{\omega} \right)$$

$$A2 = \frac{r_2 - \eta_1(r_{22} - d_{22})}{r_{22} d_{22}(\eta_1 - \eta_2)} (1 + \omega \eta_2)$$

TABLE I
Summary of Comparative-Static Results

Parameters	ω	C	μ	σ	λ	ι
y^*	+	+	-	+	+	+
Starting value	1.0	1.5	0.15	0.25	1.0	0.06
Upper value	∞	∞	10.0	3.0	∞	∞
Lower value	1.0	0.01	0.01	0.0	0.01	0.01

Notes: ω = number of undeveloped options in firm 1's portfolio; C = investment cost disadvantage of firm 1 regarding firm 2; μ = annual growth rate of the market; σ = annual sales variance faced by firm 1; λ = firm 1's risk-aversion coefficient; and ι = risk-free interest rate.

$$A3 = \xi_2 \left(1 - \theta_2 \frac{K_i}{\omega} \right).$$

It can be proved that Equations (22), (23), (24), (25), and (26) satisfy the switching condition (16). See Appendix F for the proof.

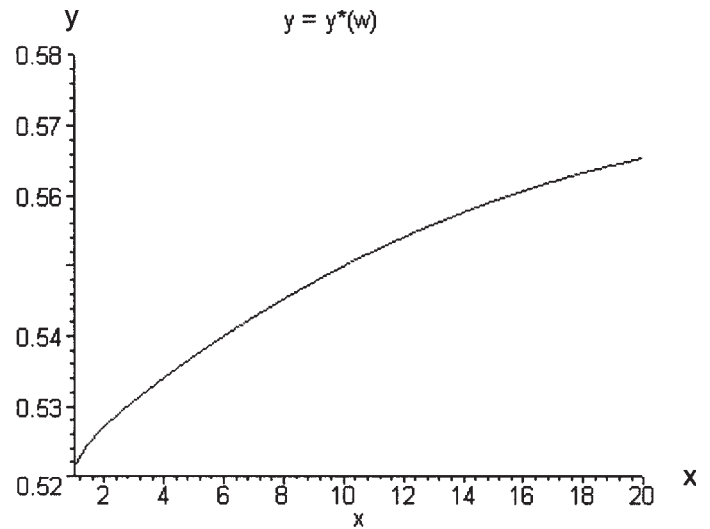
NUMERICAL ANALYSIS

This section generates a set of critical results derived from function (26). Given that y^* is the root of $H(y)$, and $dH(y)/dy$ is increasing in the vicinity of y^* , a state variable change accelerates development of full commitment investment if it either causes $H(y)$ to shift upward (y^* has to decrease to ensure that $H(y^*) = 0$), or if it causes y to increase, and thereby generates an increase in $F(y) - G(y) - \omega y G'(y) - K_i C$ along $H(y)$. Therefore, two types of effects exist: those that induce a movement on the value of y and those that induce a movement of the value of y^* and, therefore, shift the curve $H(y)$. Equation (26) shows that y^* is a function of many variables that do not affect y . In order to focus the analysis in the effect on the individual options of having a portfolio of simultaneous and related options, this study analyzes variables that only affect y^* .

The complexity of the functions makes it very difficult to obtain an analytical solution for the effect of changes in the model variables on the investment policy. This section provides, instead, a numerical analysis. Fortunately, the results are highly stable under wide ranges of parameters. The initial scenario is that of a firm with a single option, a (unit) cost disadvantage of 33 percent, and no risk aversion. The market grows at 15 percent annually with 25 percent of volatility, and the risk-free rate is 6 percent. Table 1 summarizes the effects of changes in the subset of variables upon y^* and, therefore, $H(y^*)$. It also gives the initial values used to generate the results and the range in which these results are robust, keeping the other variables at their starting values.²

It is worth noting that the model can mimic the results of traditional financial models for individual options (Black & Scholes, 1973; Merton, 1973). Similar to what occurs with

FIGURE 1
Portfolio Effect



Notes: $x \equiv \omega$ and $y \equiv H(y^*)$.

financial options, an increase in the market growth rate μ tends to accelerate development, whereas increases in market variance σ , the risk-free interest rate ι , or the risk-aversion coefficient λ , tend to delay development.

This study's central concern is on the economic effect of the presence of interdependence between simultaneous strategic investments. Therefore, the numerical analysis focuses on the marginal effect of a new strategic option on the overall value of the portfolio and on the exercise policy of the remaining options. For this purpose, it simulated the effect of new options (i.e., increase in ω) on $H(y^*)$. Figure 1 shows the geometry of the effect—the “portfolio effect.”

As Figure 1 shows, a new strategic option increases the value of the switching point at a decreasing rate. Therefore, the value added of the option, compared to the value added of a full commitment, increases at a decreasing rate. This means that, because the switching point y^* has to increase in order to ensure that $H(y^*) = 0$, an increase in ω causes development to be delayed. It can be concluded that:

1. *New partial commitments in a correlated portfolio have decreasing marginal returns; and*
2. *The greater the number of competitive options (ω), the greater are the incentives to delay developments.*

These results are particularly interesting. For an option to be worthy, it is not enough to have a positive option value by itself, but only in relation to the rest of the portfolio. In order to create value, the flexibility gain of a single option should compensate the marginal losses due to the presence of correlation with other ongoing options. These findings are consistent with similar studies in the financial literature, in particular, those of Stulz (1982) and Johnson (1987) for the contract “an option on the maximum of several assets.”

The final analysis refers to the effect of the presence of competition on the option value of the strategic investments. Competitors usually place different values on a new capital commitment, varying according to the levels of synergy between the project and the firms’ overall organizations. The expected outcome is, therefore, that the party placing the higher value on a new capital commitment first exercises the option. The variable C (cost disadvantage) captures this phenomenon. The net effect of a decrease in C is a backward shift in $H(y)$, which, in turn, causes y^* to decrease, and thereby accelerates developments. In other words, better capabilities regarding competitors overcome with the costs associated with uncertainty, and this allows the firm to exploit current opportunities. Therefore, another central result from the numerical analysis is:

3. *Firms with cost advantages prefer high-commitment investment modes.*

This result shows that real options theory supports both market power and resource-based sources of competitive advantages. Market power is the result of the oligopolistic structure of the industry. The model generates a Cournot equilibrium, which implies supracompetitive returns. These abnormal returns are generated for cases of high uncertainty and are modeled using Ito’s process. Resource-based competitive advantages come from the assumption of imperfect market. In particular, the model can reach the equilibrium having firms with different cost structures—that is, it is applicable to situations of different resources and capabilities.

DISCUSSION

This study develops a model that provides a decision rule for simultaneous and related strategic investments with a compound option value. This decision rule stresses the possibility of destroying value if the correlation between the strategic options, and between the firm and its competitors’ unit costs, are not taken into consideration. The main message is “not

every option in isolation has a positive option value that is worthy.” This highly important result for strategic management scholars and for practitioners is underemphasized in the current literature. McGrath and MacMillan (2000: 183) point out how common it is in companies to observe a failure in the design of the portfolio of strategic investments, having overlap between options in a confusing fashion.

The model has limitations. Chi (2000) shows that it is difficult to gain an accurate understanding of a particular strategic situation without explicitly modeling its structure. Stated differently, not every option model is worthy for every situation. In particular, the model developed here is only valid for a specific context: the case of a portfolio of compound options that gives the firm a first-mover competitive advantage in a market with an oligopolistic structure. The investments refer to a specific resource or capability that allows the firm to capture market growth before its competitors. In order to accomplish its objective, the model assumptions regarding the external environment are somewhat underrepresented. Market growth rates are correlated across markets, whereas competitors’ strategic decisions are only relevant within the market under analysis. Natural markets may present challenging complexities. For example, it is reasonable to expect the presence of multipoint competition between competitors. If this were the case, it would also be possible to observe the holding of the mutual forbearance hypothesis, making invalid the assumption of a Cournot equilibrium. To what extent the model’s conclusions are still valid under these circumstances is a matter for future exploration.

Another highly interesting topic to explore is whether the model holds up in the presence of a special type of capability: the ability to create and configure the optimum portfolio of strategic investments. An example is the case of a joint development of capabilities—that is, strategic alliances. If the firm has a portfolio of first-stage options, it might not have the resources or the willingness to develop all of them into second-stage options. In such a situation, the aggregate value of the firm’s undeveloped investment portfolio is magnified relative to the value of its achievable developed investment portfolio, and the firm is likely to become very sensitive to the negative effect of development on the value of its undeveloped (partial) commitments. When the firm is carrying out many strategic alliances, the negative effect may include a deterioration in business relationships between the investing firm and its partners in first-stage options. Then, the firm is even more sensitive to the negative effect of development on the value of undeveloped commitments. Therefore, firms willing to build capabilities through a network of partners should focus on their capabilities to manage portfolios of simultaneous and related growth options, as opposed to managing exclusive options within a portfolio.

CONCLUDING REMARKS

This paper investigates the trade-off between strategic flexibility and commitment for cases of multiple and simultaneous strategic investments in oligopolistic markets with high uncertainty. Previous real options models have analyzed strategic investments with independence among simultaneous investments. Whereas the independence assumption may be appropriate when investments have no interactions, it may introduce important biases in the presence of correlation between the outputs of such investments. Because interactions between strategic options diminish the marginal return of adding a new option to the portfolio, firms must evaluate new strategic commitments conditional on the existence of other ongoing commitments. Stated differently, an uncritical extension of previous real options results to situations of interactions within the portfolio can be misleading and, even worse, can encourage strategic investments that destroy economic value.

This paper demonstrates the usefulness of real option concepts in identifying sources of competitive advantage. It incorporates a strategic option that leads to a first-mover-type competitive advantage and highlights the presence of market power and idiosyncratic capabilities as a means of achieving abnormal returns. This offers a new twist to Kulatilaka and Perotti's (1998) goal of reconciling the real options and strategic approaches to the optimal use of strategic investments.

NOTES

¹ The current income y_{it} is restricted to be nonnegative, which is verified for $|e|_s \geq s_{it}$, and to be increasing in $s_{it}/|e|_s$, which is verified for $s_{it} < (|e|_s/2)$. Notice that y_{it} is an increasing function of x_s , and a decreasing function of q_s , κ_{it} , and v_{it} .

² The upper and the lower values were also analyzed changing simultaneously the value of more than one variable. As no significant modifications were found in the range, the results are not reported here.

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APPENDIX A

DERIVATION OF THE CURRENT INCOME FOR A FULL COMMITMENT INVESTMENT

It is assumed that $TVC(q_{si})$, and therefore $MC(q_{si}) = 2\kappa q_{si}$. Substituting both in Equation (3) and the expression for $MC(q_{si})$ into Equation (2), and manipulating terms, the expression for the optimal output q_{si}^* from full commitment investment si is

$$q_{si}^* = (1/\kappa_{si}) \left[P_s + (P_s/e_s) s_{si} (1 + dQ_{sj}/dq_{si}) \right]. \quad (A1)$$

From Equation (4) there are three special cases:

1. In the extreme case of $dQ_{sj}/dq_{si} = 1$, rivals fully offset production changes by a full commitment investment i , and perfectly competitive conditions hold.
2. For $dQ_{sj}/dq_{si} = 0$, firms take other firm's output levels as fixed, and standard Cournot oligopoly conditions hold.
3. For $dQ_{sj}/dq_{si} > 0$, full commitment investment si expects rivals to match, at least partially, its output adjustments. Perfect matching, $dQ_{sj}/dq_{si} = 1$, would generate perfect coordination (monopolistic collusion).

Given q_{si}^* , the income function net variable costs for full commitment investment si is given by the following expression:

$$y_{si} = \left[1 + (1/e_s) s_{si} (1 + dQ_{sj}/dq_{si}) \right] (P_s/\kappa_{si}) (P_s - AVC(q_{si}^*)). \quad (A2)$$

Notice that once the firm has incurred its fixed investment costs c_{si} , $y_{si} > 0$ is a necessary and sufficient condition for the firm to keep producing output from its investment position si . Expression (A2) can be restated as follows:

$$y_{si} = (P_s/\kappa_{si}) \left[1 - (s_{si}/e_s) (1 - dQ_{sj}/dq_{si}) \right] \left[-P_s (s_{si}/e_s) (1 - dQ_{sj}/dq_{si}) \right]. \quad (A3)$$

Assuming standard Cournot oligopoly conditions $dQ_{sj}/dq_{si} = 0$, expression (A3) becomes

$$y_{si} = (P_s/\kappa_{si}) (1 - s_{si}/e_s) (-P_s (s_{si}/e_s)). \quad (A4)$$

Aggregate demand Q_s^D within market segment s is given by the following expression:

$$Q_s^D = x_s/P_s, \quad (A5)$$

where x_s is market size in dollars. From Equation (A5), $P_s = x_s/Q_s^D$. If the market is in equilibrium, then $Q_s^D = Q_s^S$, where Q_s^S is the output aggregate supply within s . Assuming that Q_s^S is inelastic in the short run, and therefore a production capacity constraint is binding, total dollar market size x_s is determined only by the position of the Q_s^D curve. If the short-run production capacity Q_s^S is assumed to be related to the number q_s of all firms' full commitment investments within s through a constant proportionality coefficient $v > 0$, P_s can be redefined as $P_s = x_s/v_s q_s$. Substituting $x_s/v_s q_s$ into expression (A4) generates the final expression for y_{si} .

APPENDIX B

ADDITIONAL CONDITIONS OF THE CAPITAL ACCUMULATION GAME

Assume that the following inequality holds for all c_{si} :

$$c_{si} \leq (1/n_s) \left[(1/\beta_s) \sum_{j \neq i} c_{sj} \right]. \quad (B1)$$

Then there is a unique Cournot equilibrium with equilibrium price P_s , defined as

$$P_s = (1/(n_s + 1)) \left[(1/\beta_s) \sum_{j \neq i} c_{sj} \right]. \quad (B2)$$

Each firm is assumed to know its competitor's marginal cost function, and to believe that all its competitors will invest in accordance with their Cournot reaction functions. The profit function Π^i of MNE_i is specified as

$$\Pi^i(K_{si}, K_{sj \neq i}) = K_{si} \left(\beta_i - K_{si} - \sum_{j \neq i} K_{sj} - c_{si} \right), \quad (B3)$$

where K_{si} and K_{sj} are the numbers of majority investments developed by firm i and each of its competitors, respectively; and c_{sj} is the investment marginal cost of firm j . The coefficient β_s is the industry development potential in the absence of development costs. This function has two properties that are necessary for the generalization of the results to more general profit functions. First, each firm reacts negatively to capital accumulation by the others (the derivatives Π_j^i are all negative). Second, each firm's marginal value of capital decreases with the other firm's capital level ($\Pi_j^i < 0$)—that is, the capital levels are strategic substitutes.

APPENDIX C
VALUATION FUNCTIONS FOR A FULL
COMMITMENT INVESTMENT AND A
REAL OPTION POSITION

Equations (11) and (12) were derived through reproducing the return and risk characteristics of both a full commitment investment and an option position through portfolios of existing traded assets. The values of F and G must then equal the market values of these portfolios. It was assumed that the uncertainty associated with F and G follows Ito's processes. The Ito processes associated with the derivation of expressions (11) and (12) are explained at the end of this Appendix.

In general, let $F = F(y)$ be the value of a firm's opportunity to invest in a risky project, with $y = y(x, q)$. To find $F(y)$ and the optimal investment rule, consider the return of the following portfolio: hold the option, which is worth $F(y)$, and go short dF/dy units of the project. The value of this portfolio is $P = F - F_y dy$, where $F_y = dF/dy$. The short position in this portfolio will require a payment of $\delta y F_y$ dollars per time period, or no rational investor will enter the long-side of the transaction. This corresponds to the dividend stream δy times the F_y units of the project. The total return from holding the portfolio over a short time interval dt is

$$dF - F_y dy - \delta y F_y. \quad (C1)$$

If this return is risk free, then F_v must be chosen so that Equation (C1) equals the risk-free rate $r(F - F_v)dt$ —that is,

$$dF - F_y dy - \delta y F_y = r(F - F_v)dt. \quad (C2)$$

Using Ito's lemma (see below), the expression for dF is defined as

$$dF = F_y dy - \frac{1}{2} F_{yy} (dy)^2. \quad (C3)$$

Assuming that y follows a geometric Brownian motion process $dy = \mu y dt - \sigma y dz$, Equation (C3) can be rewritten as

$$dF = \mu y F_y dt + \sigma y F_y dz - \frac{1}{2} \sigma^2 y^2 F_{yy} dt. \quad (C4)$$

Finally, substituting Equation (C4) into (C2), and rearranging terms, it can be noted that the terms in dz cancel out, and the portfolio is risk free:

$$\frac{1}{2} \sigma^2 y^2 F_{yy} dt + r y F_y - r F = 0. \quad (C5)$$

The generic expression (C5) has the same structure as that in Equations (11) and (12). These expressions can be repro-

duced through following the sequences of steps described above, and substituting expression (C4) with expressions (C13) and (C14), which are derived below.

Ito's Equation for Two Ito Processes

The Black and Scholes expression for one Ito process is

$$dF = \left[F_t + a(x, t) F_x + 1/2 b^2(x, t) F_{xx} \right] dt + (x, t) F_x dz. \quad (C6)$$

An extension of expression (C6) to two Ito processes x_1 and x_2 holds the following expression:

$$\begin{aligned} dF(x_1, x_2) = & \left[F_t + a_1(x_1, x_2, t) F_{x_1} + a_2(x_1, x_2, t) F_{x_2} \right. \\ & \left. + \frac{1}{2} b_1^2(x_1, x_2, t) F_{x_1 x_1} + \frac{1}{2} b_2^2(x_1, x_2, t) F_{x_2 x_2} \right] dt \\ & + \rho_{12} b_1(x_1, x_2, t) b_2(x_1, x_2, t) \\ & + b_1(x_1, x_2, t) F_{x_1} dz_1 + b_2(x_1, x_2, t) F_{x_2} dz_2, \end{aligned} \quad (C7)$$

where $\rho_{12} = E(dz_1 dz_2) / dt$.

Since the process dq/q in Equations (11) and (12) is deterministically defined by expression (10), $b_2 = 0$. Therefore, Equation (C7) can be simplified as

$$\begin{aligned} dF(x_1, x_2) = & \left[F_t + a_1(x_1, x_2, t) F_{x_1} + a_2(x_1, x_2, t) F_{x_2} \right. \\ & \left. + \frac{1}{2} b_1^2(x_1, x_2, t) F_{x_1 x_1} \right] dt + b_1(x_1, x_2, t) F_{x_1} dz_1. \end{aligned} \quad (C8)$$

APPENDIX D
MARKET VALUE OF A FULL COMMITMENT
INVESTMENT POSITION

Equations (17), (13), and (14) have the unique solution

$$F(y) = \begin{cases} a_1 y^{s_1} + a_1' y^{s_1'} + \xi_1 y & \text{if } 0 \leq y \leq y^* & (D1) \\ a_2 y^{s_2} + a_2' y^{s_2'} + \xi_2 y & \text{if } y^* \leq y \leq \infty, & (D2) \end{cases}$$

with $Zy = y/d$, where $a_1' = a_2' = 0$, and

$$\begin{aligned} s_1 = & \frac{1}{2} - (r_1 - d_{11}) / \sigma^2 \\ & + \left[\left((r_1 - d_{11}) / \sigma^2 - 1/2 \right)^2 + 2r_1 / \sigma^2 \right]^{1/2} > 1 \end{aligned}$$

$$s_2 = \frac{1}{2} - (r_1 - d_{12})/\sigma^2$$

$$-\left[\left((r_1 - d_{12})/\sigma^2 - 1/2\right)^2 + 2r_1/\sigma^2\right]^{1/2} < 0$$

$$r_1 - d = \begin{cases} \rho - K_2 & \text{if } 0 \leq y \leq y^* \\ \rho - K_2 - K_i & \text{if } y^* \leq y \leq \infty \end{cases}$$

$$r_1 = \iota$$

$$d = \begin{cases} d_{11} = \iota - \rho + K_2 & \text{if } 0 \leq y \leq y^* \\ d_{12} = \iota - \rho + K_2 + K_i & \text{if } y^* \leq y \leq \infty \end{cases}$$

$$\xi_1 y = y/d_{11}, \text{ and } \xi_2 y = y/d_{12}.$$

Notice that $\xi_1 y$ approaches zero as y approaches zero, whereas $\xi_2 y$ is a perpetuity corresponding to the flow of revenue from the D -position discounted at the risk-adjusted discount rate $\iota - K_2 - K_i$, but which is also expected to grow at the rate $\rho = \mu - \lambda\sigma$. It can be shown that $\xi_1 y$ is the upper bound for F , and $\xi_2 y$ is its lower bound (Dixit & Pindyck, 1994; Williams, 1992). Furthermore, $a_1 y^{s_1}$ is the value of the opportunity to develop a D -position in the future, if the income y increases above its current level, whereas $a_2 y^{s_2}$ is the value of the opportunity to divest a D -position if the project income decreases below its current level.

The values of a_1 and a_2 in Equations (D1) and (D2) can be obtained by applying the conditions of continuity of the F function at y^* (Dixit & Pindyck, 1994). By equating (D1) and (D2), the following expressions for a_1 and a_2 are obtained:

$$a_1 = \left[r_1 - s_2(r_1 - d_{11}) \right] / \left[r_1 d_{11} (s_1 - s_2) y^{*(1-s_1)} \right] \quad (D3)$$

and

$$a_2 = \left[r_1 - s_1(r_1 - d_{12}) \right] / \left[r_1 d_{12} (s_1 - s_2) y^{*(1-s_2)} \right]. \quad (D4)$$

After substituting Equations (D3) and (D4) into Equations (D1) and (D2), the following expression for $F(y)$ is obtained.

$$F(y) = \begin{cases} \left[r_1 - s_2(r_1 - d_{11}) \right] / \left[r_1 d_{11} (s_1 - s_2) \right] y^* (y/y^*)^{s_1} + \xi_1 y & \text{if } 0 \leq y \leq y^* \\ \left[r_1 - s_1(r_1 - d_{12}) \right] / \left[r_1 d_{12} (s_1 - s_2) \right] y^* (y/y^*)^{s_2} + \xi_2 y & \text{if } y^* \leq y \leq \infty. \end{cases}$$

APPENDIX E

MARKET VALUE OF AN OPTION POSITION

Because the underlying differential equations have the same structure, the procedure for determining the market value of an undeveloped position $G(y)$ is similar to that followed to solve for $F(y)$. Equations (19), (13), and (15) have a unique solution:

$$G(y) = \begin{cases} b_1 y^{\eta_1} + b_1' y^{\eta_1} + \theta_1 \Omega_1 & \text{if } 0 \leq y \leq y^* \\ b_2 y^{\eta_2} + b_2' y^{\eta_2} + \theta_2 \Omega_2 & \text{if } y^* \leq y \leq \infty, \end{cases} \quad (E1)$$

$$(E2)$$

with $Zy = y/d$, where $b_1' = b_2'$ from Equation (13), and

$$\eta_1 = \frac{1}{2} - (r_{21} - d_{21})/\sigma^2$$

$$+ \left[\left((r_{21} - d_{21})/\sigma^2 - 1/2 \right)^2 + 2r_{21}/\sigma^2 \right]^{1/2} > 1$$

$$\eta_2 = \frac{1}{2} - (r_{22} - d_{22})/\sigma^2$$

$$- \left[\left((r_{22} - d_{22})/\sigma^2 - 1/2 \right)^2 + 2r_{22}/\sigma^2 \right]^{1/2} < 0$$

$$r_2 - d = \begin{cases} \rho - K_2 & \text{if } 0 \leq y \leq y^* \\ \rho - K_2 - K_i & \text{if } y^* \leq y \leq \infty \end{cases}$$

$$r_2 = \begin{cases} r_{21} = \iota & \text{if } 0 \leq y \leq y^* \\ r_{22} = \iota + K_i/\omega & \text{if } y^* \leq y \leq \infty \end{cases}$$

$$d = \begin{cases} d_{21} = \iota - \rho + K_2 & \text{if } 0 \leq y \leq y^* \\ d_{22} = \iota - \rho + K_2 + K_i(1 + 1/\omega) & \text{if } y^* \leq y \leq \infty \end{cases}$$

$$\theta_1 \Omega_1 = 0$$

and

$$\theta_2 \Omega_2 = K_i/\omega(F - K_i)/d_{22}$$

with

$$\Omega_2 = K_i/\omega(F - K_i).$$

Notice the similarities between $\xi_2 y$ (defined above, expression (D2)) and $\theta_2 \Omega_2$. The latter is a perpetuity corresponding to the flow of revenue from the capital gain per U -position resulting from producing one D -position, which is discounted

at the risk-adjusted discount rate $d_{22} + \rho$, but which is also expected to grow at the rate $\rho = \mu - \lambda\sigma$. Furthermore, $\theta_1\Omega_1$ is the upper bound for G , and $\theta_2\Omega_2$ is its lower bound (Dixit & Pindyck, 1994; Williams, 1992).

The values of b_1 and b_2 in Equations (E1) and (E2) can be obtained by applying the conditions of continuity of the F function at y^* . By equating (E1) and (E2), the following expressions for b_1 and b_2 are obtained:

$$b_1 = \left[r_{21} - \eta_2(r_2 - d_{21}) \right] / \left[r_{21}d_{21}(\eta_1 - \eta_2)y^{*(1-\eta_1)} \right] \quad (\text{E3})$$

and

$$b_1 = \left[r_{22} - \eta_1(r_{22} - d_{22}) \right] / \left[r_{22}d_{22}(\eta_1 - \eta_2)y^{*(1-\eta_2)} \right]. \quad (\text{E4})$$

After substituting Equations (E3) and (E4) into (E1) and (E2), the following expression for $G(y)$ is obtained:

$$G(y) = \begin{cases} \left[r_2 - \eta_2(r_2 - d_{21}) \right] / \left[r_2d_{21}(\eta_1 - \eta_2) \right] y^* (y/y^*)^{\eta_1} \\ \quad + \theta_1\Omega_1 & \text{if } 0 \leq y \leq y^* \\ \left[r_2 - \eta_2(r_2 - d_{21}) \right] / \left[r_2d_{22}(\eta_1 - \eta_2) \right] y^* (y/y^*)^{\eta_2} \\ \quad + \theta_2\Omega_2 & \text{if } y^* \leq y \leq \infty. \end{cases}$$

APPENDIX F PROOF FOR EQUATION (26)

The function H in Equation (20) satisfies to corner conditions $H(0) < 0 < H(\infty)$. The latter inequality holds under the assumption $\rho < \iota$. Also, at y^* , it satisfies $y^*H'(y^*) = K_i C \eta_1 - [(\eta_1 - 1)\xi_1 + (s_1 - \eta_1)\chi_1]y^*$, from Equations (22), (23), (24), and (25), with $\chi_1 = (r_1 - s_2(r_1 - d_{11})) / (r_1 - d_{11}(s_1 - s_2)) > 0$.

Because the term in brackets is positive, H is increasing at y^* if the following inequality holds true:

$$y^* < K_i C \eta_1 / [(\eta_1 - 1)\xi_1 + (s_1 - \eta_1)\chi_1].$$

In this case, it is sufficient to show that Equation (26) can be the only root of function H on the interval $[0, \infty]$. On the interval $[0, y^*]$, the function H is strictly concave, given Equations (22), (23), (24), and (25). Because $H'(y^*) > 0$, there is no root below y^* . On the upper interval $[y^*, \infty]$, the function H is bounded below:

$$H(y) > \xi_2 y - G(y) - \omega y G'(y) - CK_i \equiv B(y).$$

The function B is convex on an initial interval (y^*, y^0) , and concave on the subsequent interval (y^0, ∞) , with an inflexion point satisfying $y^* < y < \infty$. Also, B satisfies $B'' < H''$ for all $y^* < y < \infty$. Given this lower bound B and the initial condition $H'(y^*) > 0$, there can be no root on the upper interval (y^*, ∞) if $B(y)/y \geq 0$ as $y \rightarrow \infty$. With Equations (24), (25), and (20), this result holds if $\rho < \iota$, as assumed.