Instructions

Answer all parts of all three questions. **Be sure to allocate your time appropriately.**

Show your work clearly and explain your answers briefly but in enough detail that your reasoning can be followed. No credit will be given for answers without appropriate reasoning. Do only the work requested. **Be sure your answer is organized in a way that can be followed by the graders.**

If you become stuck on a part of a problem or make an algebra error that prevents you from solving a part of a problem, explain how you would complete the rest of the problem. Be very specific.

**To avoid loss of clarity when your answers are photocopied for the graders, be sure to use dark pencil or pen, write clearly, and do not write near the edge of the paper.**
A profit-maximizing firm produces two outputs using two inputs, capital and labor, with input prices \( r \) per unit for capital and \( w \) per unit for labor. The process has joint products, so that when the input bundle is \((K, L)\), the output is simultaneously \( x = f(K, L) \) units of the first output and \( y = g(K, L) \) units of the second output. In the relevant region, \( g \) is concave with strictly positive first partials while \( f \) is concave with \( f_K \) strictly positive but \( f_K \equiv f_{KL} \equiv f_{LK} \equiv 0 \), where subscripts indicate partial derivatives.

The firm must sell everything it produces. The firm is a monopolist in the first market, facing inverse demand \( p = P(x) \), where \( P' < 0 \). The second market is competitive, with price \( q \) per unit.

The firm faces a cash constraint in the current period: inputs cannot cost more than \( C \) in total.

a. Set up the firm's optimization problem and find the first-order conditions for a solution at which both inputs are used in strictly positive amounts.

b. In your first-order conditions from part (a), are the multipliers necessarily nonzero? Interpret the first-order conditions.

c. [For this part, assume the multipliers in your first-order conditions are nonzero, if they aren't necessarily so.] Starting from a solution at which both inputs are used in strictly positive amounts, suppose \( q \), the market price for the second output, increases. What happens to the \textit{price} of the monopoly good?
2. A single-product, risk-neutral, expected-profit maximizing monopolist faces aggregate demand \( Q = 150 - 2p \) where \( Q \) is the aggregate quantity and \( p \) is the per-unit price. The firm has constant marginal and average cost, but the level of that unit cost is a random variable taking value 20 with probability 0.5 and taking value 30 with probability 0.5. The firm knows the distribution of its unit cost, but it does not know the actual unit cost when it makes its choice.

a. Find the expected-profit maximizing solution and the expected profit level.

b. Suppose the firm could hire an expert to forecast its unit cost before making its decision. The forecast is always 20 or 30. If the true unit cost were 20, the expert’s predictions are known to be correct 80% of the time, but if the true unit cost were 30, the expert’s predictions are known to be correct only 60% of the time.

   What is the maximum the firm would be willing to pay to hire the expert? Does the firm’s expected output/price differ from that in part (a)?

c. Assume the firm has hired the expert and is committed to its production/pricing decisions as determined in part (b). Suppose the firm has an accounting book value and it is subject to rate-of-return regulation based on that value. If the firm’s payoff exceeds the allowed multiple of its book value, then all the “excess profit” must be paid as tax.

   Consider three different ways the regulation could be applied.

   First version: The regulator knows the \textit{a priori} cost distribution, but not the expert’s forecast or the true cost. When the regulator observes the firm’s production/pricing choice, it computes expected profit based on the regulator’s information (i.e., without knowing the expert’s forecast or the actual cost), and applies the tax to this expected profit.

   Second version: The regulator waits until the actual cost is observed, and applies the tax based on the realized profit.

   Third version: The regulator knows both the \textit{a priori} cost distribution and the expert’s accuracy and it observes the expert’s forecast. When the regulator observes the firm’s production/pricing choice and the expert’s forecast, it computes expected profit based on the regulator’s information (i.e., without knowing the actual cost, but with the same information as the firm), and applies the tax to this expected profit.

   Before the firm obtains the expert’s forecast, would it care which version of the regulation would be applied? Explain.
3. A price-taking profit-maximizing firm produces a single output using \( n \) inputs according to production function \( f(x) \), where \( x \) is the vector of inputs and \( f(0) = 0 \). It faces prices \( p \) per unit for its output and \( q_i \) per unit for input \( i \), \( i = 1, 2, \ldots, n \). Its profit function, \( \pi(p, q) \), is defined as the maximum of \( pf(x) - q \cdot x \), where the maximization is taken with respect to the input vector \( x \). Consider the region of prices for which a maximizer exists.

a. Show \( \pi(p, q) \) is homogeneous of degree one in prices [define HD1 as part of your answer].

b. Show \( \pi(p, q) \) is convex in prices [define convex as part of your answer].

c. Show \( \pi(p, q) \) is non-decreasing in \( p \).

d. Show \( \pi(p; q) \) is non-increasing in each \( q_i \).

e. Show \( \pi(p, q) \) is nonnegative.