How Do Solvers Work?

- Linear Programming (LP) is cut and dried
  - The Simplex method is a universally reliable and available approach to the problem
  - Formulation is the only issue
- Why bother with Nonlinear Programming (NLP)?
  - At the aggregate level, we don't observe the discrete jumps associated with LP basis changes – **Realism**

Numerical Methods

- Numerical methods for NLP are available in many highly distinct flavors
  - There are both good and bad methods in use
  - It is important to know how they work because
    - When they fail, understanding why helps tell you know what to do about it
    - You can avoid un-necessary failures

Numerical Methods – Easy Problems

- The simplest problem:

$$\underset{x}{\text{minimize}} \quad F(x)$$

- or *almost* equivalently:

$$\nabla F(x) = \left[\frac{\partial F(x)}{\partial x_i} \right]_{i=1,\ldots,n} = 0$$

- Why are these not *really* equivalent?

Numerical Methods (cont'd.)

- Virtually all methods that are not algebraic are iterative, generating a sequence of points:

$$\underline{x}^{0}_{0}, \quad \underline{x}^{1}_{1}, \quad \underline{x}^{2}_{2}, \quad \ldots, \quad \underline{x}^{k}_{k}, \quad \underline{x}^{k+1}_{k+1}, \quad \ldots$$

- A good method is one where
  - The sequence of gradient values converges to zero
  - And does so "quickly"

Numerical Methods (cont'd.)

- Ideally, one can prove a method's convergence
- Some methods in use by applied economists are not guaranteed to converge:
  - Gauss-Seidel iteration
  - Jacobi iteration
  - Newton's method

Numerical Methods (cont'd.)

- Gauss-Seidel and Jacobi Iteration are minor variants of the following scheme:
  - Rearrange so that it is equivalent to:
  - Then use the *iterative* scheme: $k+1$ k
  - Convergence varies by case – and may not occur!

Numerical Methods (cont'd.)

- Newton's Method is based on approximation:

$$\nabla F(x) \approx \nabla F(x^k) + \nabla^2 F(x^k)^t (x - x^k)$$

  - Setting the right-hand side of this equal to zero and solving for *x* gives us:

$$x^{k+1} = x^k - \nabla^2 F(x^k)^{-1} \nabla F(x^k)$$

  - This method is also unreliable and may even break down

Numerical Methods (cont'd.)

- Methods that can be proved to converge for problems with mild restrictions do the following:
  - At each iteration they compute a search direction that is guaranteed to result in an objective improvement
  - They use a steplength procedure to determine how far to go in the search direction and verify that progress is made

Numerical Methods (cont'd.)

- Good *implementations* of good methods:
  - Apply high quality convergence conditions
    - Typically the gradient must be "close enough" to 0
    - Lack of progress is not good enough
  - Are designed to work even for difficult problems, but are also fast for easy problems

Numerical Methods (cont'd.)

- Candidates for Good Methods for Computing Search Directions (in order of reliability):
  - Newton's method if protected against indefinite or singular Hessians
  - Quasi-Newton methods
  - Conjugate gradient methods (usually only applied to quite large problems)

Numerical Methods (cont'd.)

- What about constrained problems?
- The simplest case:

$$\underset{x}{\text{minimize}}\ F(x) \quad \text{subject to: } x \geq l$$

- or, *almost* equivalently

$$\begin{array}{ccc} \nabla F(x) - \lambda = 0,\ \lambda \geq 0, & & \nabla F(x) \geq 0,\ x \geq l \text{ and} \\ & \text{or} & \\ x \geq l \text{ and } \lambda^t (x - l) = 0 & & (x - l)^t \nabla F(x) = 0 \end{array}$$

Numerical Methods (cont'd.)

- New issues with this problem:
  - Which constraints are active?
  - What are the values of the Lagrange multipliers?

Numerical Methods (cont'd.)

- Bad methods – Once a constraint becomes active it remains active forever
- Good methods
  - Use a dynamic active set strategy adding and deleting constraints from the active set during iterations
  - Modify the steplength procedure to ensure that as long as we start feasible, we stay feasible

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Numerical Methods (cont'd.)

- What about more complicated constraints?

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- or,

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- (Notice we are only dealing with equality constraints at this point)

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Numerical Methods (cont'd.)

- Given any feasible point x^0, this problem can be transformed with linear algebra to an equivalent problem in a lower dimension space:

$$\underset{y}{\text{minimize}}\ F(x^0 + Zy)$$

  - If A has m rows, then the dimension of y is $n\text{–}m$, and Z has n rows and $n\text{–}m$ columns
  - Does this make sense?

Numerical Methods (cont'd.)

- Yes! It makes sense!
  - Each of the linear constraints effectively removes a degree of freedom from the solution
  - With n variables and m linear constraints, we should end up with $n\text{–}m$ degrees of freedom
  - (Details for this argument can be found in Gill, Murray and Wright)

Numerical Methods (cont'd.)

- So, with this class of problems, the same things matter:
  - The search direction must still be a direction of decrease of the objective, and
  - A steplength procedure that verifies progress in each iteration must be used
- In addition, careful attention must also be paid to the linear algebra used to perform the transformation

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Numerical Methods (cont'd.)

- What about even more complicated constraints?
- or,

$$\nabla F(x) - A^t \lambda = 0, \; Ax \geq b,$$

- (Notice all constraints are inequalities at this point)

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Numerical Methods (cont'd.)

- Good methods combine the ideas from methods for bounded variables and linear equality constraints
  - A dynamic active set strategy
  - Estimates of Lagrange multipliers
  - Linear algebra is used to (dynamically) reduce the space for the search

Numerical Methods (cont'd.)

- Summary:
  - Classes of problems examined so far are "not terribly hard"
  - Iterative methods have a number of identifiable features that make them reliable
  - Maintaining feasibility is not hard
  - What kinds of problems have we neglected so far?

Numerical Methods (cont'd.)

- Fully general equality constraints

$$\underset{x}{\text{minimize}}\ F(x) \ \text{ subject to}: \ C(x) \geq 0$$

- or,

$$\nabla F(x) - \nabla C(x)^t \lambda = 0,$$
$$C(x) \geq 0, \lambda \geq 0, \text{ and } \lambda^t C(x) = 0$$

Numerical Methods (cont'd.)

- These problems are much tougher than the preceding problems because feasibility cannot be maintained in general
  - Methods must trade off improvements in the objective against constraint violations
  - Search directions will generally not be feasible even for small steps

Numerical Methods (cont'd.)

- There are more methods for problems with nonlinear constraints
  - These represent philosophical differences regarding the importance of objective improvement versus infeasibility during the iterative process
  - No method is universally better
  - All are based on linearly constrained subproblems – so, a good method will implement these well

Numerical Methods (cont'd.)

- Even with good methods, it may be difficult to solve these problems
  - **Important fact** – Problems with nonlinear constraints are harder than linearly constrained (or unconstrained) problems
  - Problems with nonlinear constraints **should be avoided** if possible

Numerical Methods (cont'd.)

- These problems can often be avoided by judicious formulation
  - Ex. 1:

$$\begin{aligned}&\underset{x}{\text{minimize}}\ 0\\&\text{subject to: } C(x)=0\end{aligned} \quad \text{or} \quad \underset{x}{\text{minimize}}\ C(x)^t C(x)$$

Numerical Methods (cont'd.)

- Ex. 2:

$$\underset{x,y}{\text{minimize}}\ F(x) \quad \text{or} \quad \underset{y}{\text{minimize}}\ F[C(y)]$$