

Multivariate Functions (Cont'd.)

- A final case

$$F(x) = \alpha \left[\sum_{i=1}^n \beta_i x_i^{-\rho} \right]^{-1/\rho} = \alpha [\beta_1 x_1^{-\rho} + \beta_2 x_2^{-\rho} + \dots + \beta_n x_n^{-\rho}]^{-1/\rho}$$

- and $\sigma < 0$
- Used to describe production possibilities
- Called the Constant Elasticity of Transformation function, or CET

Multivariate Functions (Cont'd.)

- CES is increasing in each of its arguments and concave (strictly only in the case of decreasing returns to scale)
- CET is also increasing in each of its arguments, but is convex
- If $\alpha=1$, $\beta_i=1$, and $\rho=-2$ (a CET case), is this function familiar?

$$2 \quad 2 \quad 2 \quad 1/2$$

Multivariate Functions (Cont'd.)

- Multivariate linear response and plateau

$$F(x) = \min [\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n, \beta]$$

where the α_i and β are constants.

Multivariate Functions (Cont'd.)

- Like the Leontief, this function can be simulated by the objective of a linear program:

$$F(x) = \text{maximize } y$$
$$\text{subject to: } y \leq \alpha^i x^i, \quad i = 1, 2, \dots, n$$

Composite Functions and Nesting

- Notice the relationship between the univariate linear response and plateau, the multivariate Leontief, and the multivariate linear response and plateau. Can we write these in the form $F(x)=G[H(x)]$? (Identify F , G , and H .)
- The general name for this type of operation is called the composition of functions, and the result is called a “composite function.”

Composite Functions (Cont'd.)

- Composition is how we get complicated functional forms from a small number of simple functions
- Consider the flexible functional forms called “generalized quadratics”
 - Translog
 - Generalized Leontief
 - Fourier

Composite Functions (Cont'd.)

- These are simply composites of the multivariate quadratic with other functions – e.g. for the translog,

$$F(x) = \alpha_o + \sum_{i=1}^n \alpha_i \ln(x_i) + \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n \beta_{ij} \ln(x_i) \ln(x_j)$$

- This could be written $F(x) = Q[\ln(x_1), \ln(x_2), \dots, \ln(x_n)]$ where $Q()$ is the multivariate quadratic.

Composite Functions (Cont'd.)

- Other instances of composite functions include:

- CES (three-fold: constant elasticity, multivariate linear, and constant elasticity again)
- Stone-Geary or linear expenditure system (affine and Cobb-Douglas)

$$F(x) = \alpha \prod_{i=1}^n (x_i - \gamma_i)^{\beta_i}$$

- Nested CES (see below)

Nested Functions

- A particular type of composite function is called a “nested functional form”
- Nesting is typically used to define production relationships via production functions or preferences via utility functions
- With nesting, the arguments of a production function may be production functions themselves

Nested Functions (Cont'd.)

- Nesting can (and often does) go on for several levels
- Consider a two level example (due to Manne) in the case of a production function that produces “GDP” from capital (K), labor (L), non-electric energy (N) and electric energy (E)

Nested Functions (Cont'd.)

$$F(K, L, E, N) = \alpha \left[\beta_1 (K^\delta L^{1-\delta})^{-\rho} + \beta_2 (E^\gamma N^{1-\gamma})^{-\rho} \right]^{1/\rho}$$

- Notice that K and L are combined using a Cobb-Douglas relationship, and E and N are also combined using a different Cobb-Douglas
- The $K&L$ and $E&N$ aggregates are then combined using a CES relationship

Nested Functions (Cont'd.)

- CES functions are commonly used functions for nesting
- Note that elasticities of substitution across nests are generally not constant even when the nesting functions are CES

Convexity of Composite Functions

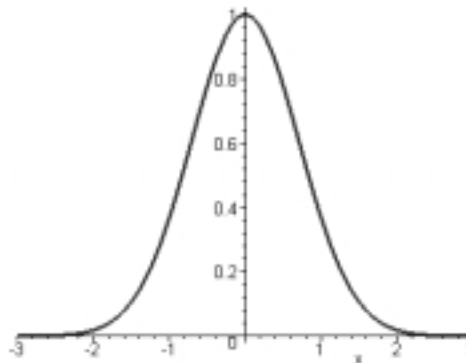
- Convexity is an important determinant of the behavior of the optimization problems lying behind economic theory
- Is convexity inherited by composite functions?
 - I.e., if $f(y)$ is convex and $g(x)$ is convex, is $f[g(x)]$ convex?

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Convexity of Composite Functions (Cont'd.)

- Let $f(y) = \exp(-y)$ and $g(x) = x^2$. Is the composite function $f[g(x)]$ convex?
- $f[g(x)] = \exp(-x^2)$.
- Is this a familiar function?
- What does its graph look like?
- Convex?



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Convexity of Composite Functions (Cont'd.)

■ Sufficient conditions to guarantee that a composite function will be convex:

- If $f(y)$ and $g_i(x)$ are convex functions, and if $f(y)$ is increasing in each of its arguments, then the composite function $f[g_1(x), g_2(x), \dots, g_m(x)]$ is convex.

Convexity of Composite Functions (Cont'd.)

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Convexity of Composite Functions (Cont'd.)

- Recall that the two input CES is concave
 - Can we use this fact and the results above to show that the n input CES is concave?
 - Can we use these results to show that the nested CES is concave?

Implicit Relationships

- An important and growing approach to describing relationships is based on implicit functions
 - The relationship between a set of variables (e.g., x and y) cannot be written explicitly (e.g., as $y = f(x)$), but must be written implicitly (e.g., as $g(x,y) = 0$)
 - E.g., $x^2 + y^2 = z$ cannot be written in the form $y = f(x,z)$ or $z = f(x,y)$

Convex Programming

- “Convex programming” refers to a special class of problems that are particularly nice from both modeling and computational perspectives
- Important:
 - *Convex programming problems have exactly one optimal objective value*
 - *Sometimes convex programming problems have exactly one optimal solution*

Convex Programming (Cont'd.)

- Uniqueness of the optimal objective value is important because:
 - From a modeling perspective it means that changes in the objective value for a convex programming problem can be attributed to changes in the model
 - From a computational perspective, if we find a locally optimal solution, we have found *the* globally optimal value of the objective

Convex Programming (Cont'd.)

- Convex programming problems have these properties:
 - The feasible region is a convex set, and
 - If the goal is minimization (maximization), the objective function is convex (concave) over the feasible region

Convex Programming (Cont'd.)

- Consider the problem:

$$\text{minimize } F(x)$$

subject to :

$$g^i(x) \geq 0 \quad i = 1, \dots, t, \text{ and}$$

where x is an n -dimensional vector of problem variables.

Convex Programming (Cont'd.)

- The following conditions are sufficient to guarantee that the feasible region is convex:
 - $g_j(x)$ is concave for $j=1,2,\dots,t$, and
 - $g_j(x)$ is linear for $j = t+1,t+2,\dots,m$.

Convex Programming (Cont'd.)

- Again, these conditions are sufficient, but not necessary. For instance, the feasible region is still convex if:
 - $g_j(x)$ may be concave for $j=1,2,\dots,t$ only over the region where the other constraints are satisfied, or
 - $g_j(x)$ may be “equivalent” to linear functions for $j = t+1,t+2,\dots,m$.

Convex Programming (Cont'd.)

- A related type of problem can be defined — the *strictly convex program*
 - A convex program with a *strictly convex* (concave) objective to be minimized (maximized) is called a *strictly convex program*

Convex Programming (Cont'd.)

- A strictly convex program has
 - Unique objective function value, and
 - Unique optimal variable values x
 - I.e., *any* optimal solution is *the* unique solution
 - Added restriction involves only the objective
 - Objective must be strictly convex with respect to *all* problem variables

Lagrange Multipliers: Interpretation and Signs

- *Universal* interpretation of Lagrange multipliers is “marginal units of the objective per marginal unit of the right-hand side of the constraint”
 - Ex. #1 if the objective is in dollars, and the constraint states that acres of land used cannot exceed the acres available,
 - Then the Lagrange multiplier on the constraint is in units of dollars per acre of land available

Lagrange Multipliers (Cont'd.)

- Ex. #2 if the objective is in florins, and the constraint states that the cases of tomatoes at warehouse #1 next week shipped out cannot exceed the cases of tomatoes shipped in to warehouse #1 cumulatively,
- Then the Lagrange multiplier is in units of florins per case of tomatoes shipped in cumulatively

Lagrange Multipliers (Cont'd.)

- Objective may not be in convenient units (e.g., utility)
 - Unit analysis used to convert Lagrange multipliers to monetary units
 - Ex. regardless of the units of the objective, if I have the multipliers from a constraint on initial wealth in dollars (X) and a constraint on land (Y),
 - Then I can find the value of land in dollars by taking the ratio Y/X (What are the units?)

Lagrange Multipliers (Cont'd.)

- Care is needed in interpreting the resulting units
- It is important to select the constraint for the denominator such that it will always be an active constraint (otherwise we would be dividing by zero)

Lagrange Multipliers (Cont'd.)

- Consider the signs of Lagrange multipliers
 - The signs of Lagrange multipliers are critically important because –
 - | They define economic value, and
 - | They can indicate failure of optimality conditions

Lagrange Multipliers (Cont'd.)

- What do the signs of Lagrange multipliers for an optimal solution depend upon?
 - The direction of optimization (max or min), and
 - The nature of the relationship between right- and left-hand side of the constraint

Lagrange Multipliers (Cont'd.)

- Signs of Lagrange multipliers for an optimal solution:

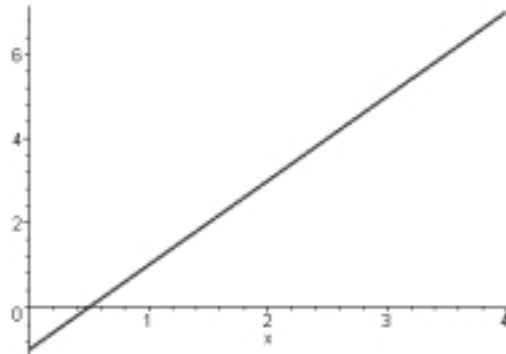
	Maximization	Minimization
\leq	+	-
\geq	-	+
$=$?	?

Lagrange Multipliers (Cont'd.)

- Q: How do you remember this?
- A: The Lagrange multiplier measures the change in the optimal objective value associated with an increase in the right-hand side of the constraint.
- Why not draw a picture?

Lagrange Multipliers (Cont'd.)

- Maximize $f(x)$ subject to: $x \leq 3$

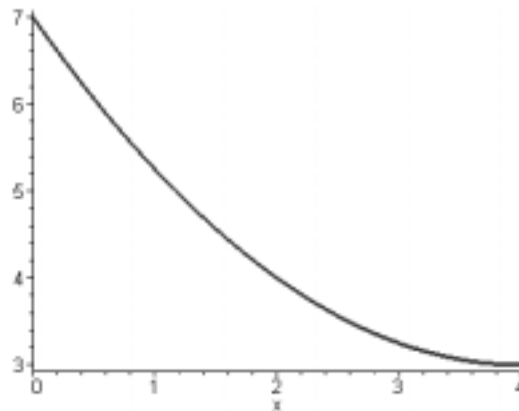


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Lagrange Multipliers (Cont'd.)

- Minimize $f(x)$ subject to: $x \leq 3$

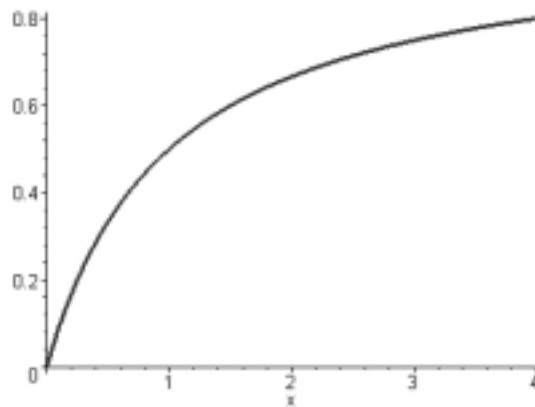


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Lagrange Multipliers (Cont'd.)

- Minimize $f(x)$ subject to: $x \geq 1$



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