Mathematical Fundamentals

- Numbers, Vectors, and Such
  - Our focus is on real numbers
  - Vectors are ordered collections of numbers
    - An *n*-vector is an ordered collection of *n* numbers
    - C*omponents* of a vector by adding a subscript

Math Fundamentals (Cont'd.)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \left[y_i \right]_{i=1,\cdots,n} \quad z_j^k = \begin{bmatrix} z_{j1}^k \\ \vdots \\ z_{jm}^k \end{bmatrix}$$

Math Fundamentals (Cont'd.)

- Vectors can be added and subtracted (if they are of the same dimension) and multiplied by scalars (single numbers)

- Vectors can act as matrices and be multiplied by matrices (but only if they are *conformable)*

(Math Fundamentals cont'd.)

- Convexity — Sets

  - A ***convex combination*** of two vectors (of equal dimension) means:

$$\lambda x + (1-\lambda) y = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} + (1-\lambda) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \lambda x_1 + (1-\lambda) y_1 \\ \lambda x_2 + (1-\lambda) y_2 \\ \vdots \end{bmatrix}$$

where λ is a scalar between zero and one

(Math Fundamentals cont'd.)

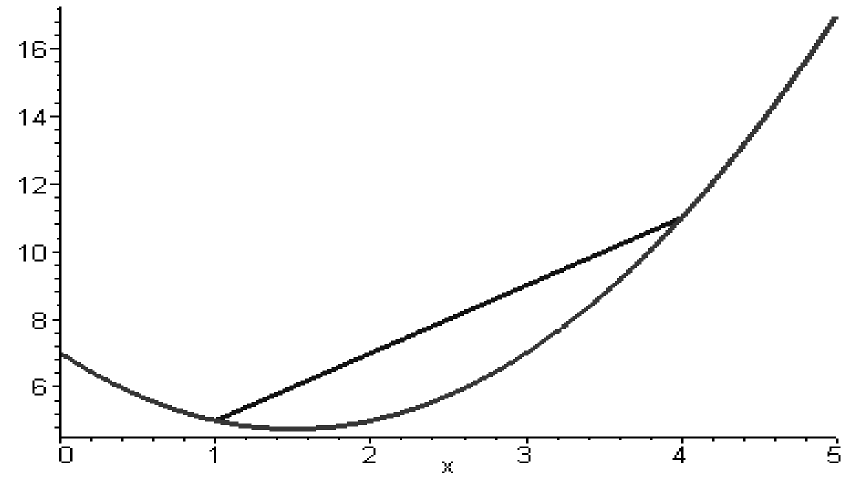
- A ***set*** is a collection of elements. We will be interested in sets whose elements are typically made up of vectors of a fixed dimension and uncountable.

- A set *S* is a ***convex set*** if and only if for every two points *x* and *y* in *S*, every convex combination of *x* and *y* is also in the set.

(Math Fundamentals cont'd.)

- Convexity — Functions
  - Functions are recipes the produce a single value for each point in the function's domain
  - A function is a ***convex function*** over its domain if and only if for every pair of points *x* and *y* in the domain of *f* and every scalar λ, $f(\lambda x+(1- \lambda)y) \leq \lambda f(x)+(1- \lambda)f(y)$.

(Math Fundamentals cont'd.)



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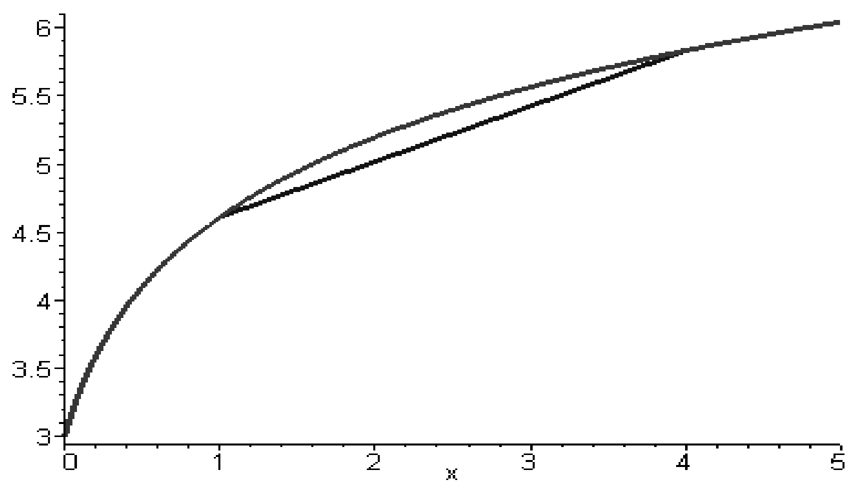
(Math Fundamentals cont'd.)



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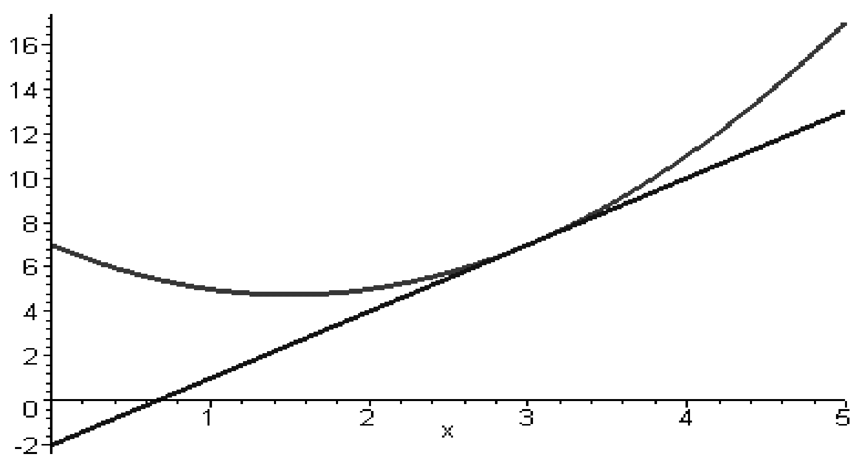
(Math Fundamentals cont'd.)

- Two other definitions that are equivalent to the above when *f*() is once or twice differentiable:
  - *f* is convex if for every *x* and *y* in the domain of *f*,

$$f(y) \geq f(x) + (y - x) f'(x).$$

(Math Fundamentals cont'd.)



(Math Fundamentals cont'd.)

- *f* is convex if for every *x* in the domain of *f*,

$$f''(x) \geq 0.$$

(Math Fundamentals cont'd.)

- "Strict convexity" that has similar definitions. These are as follows:
  - A function is a ***strictly convex function*** over its domain if and only if for every pair of distinct points *x* and *y* in the domain of *f* and every scalar λ strictly between zero and one, $\lambda f(x+(1-\lambda)y) < \lambda f(x)+(1-\lambda)f(y)$.

(Math Fundamentals cont'd.)

- *f* is strictly convex if for every distinct *x* and *y* in the domain of *f*,

$$f(y) > f(x) + (y - x) f'(x).$$

(Math Fundamentals cont'd.)

- *f* is strictly convex if for every *x* in the domain of *f*,

- A function is (strictly) concave if the negative of the function is (strictly) convex.

(Math Fundamentals cont'd.)

- These concepts generalize to the multivariate case in a straightforward manner.
  - A function is a ***convex*** over its domain if and only if for every pair of points *x* and *y* in the domain of *F* and every scalar λ between zero and one,

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y)$$

(Math Fundamentals cont'd.)

- *F* is convex if for *x* and *y* in the domain of *F*,
- *F* is convex if for every *x* in the domain of *F*,

2

is positive semi-definite.

(Math Fundamentals cont'd.)

- Again similar to the univariate case, a function is a ***strictly convex*** over its domain if and only if for every pair of distinct points *x* and *y* in the domain of *F* and every scalar λ strictly between zero and one,

$$F(\lambda x + (1-\lambda)y) < \lambda F(x) + (1-\lambda)F(y)$$

(Math Fundamentals cont'd.)

- *F* is strictly convex if for every distinct *x* and *y* in the domain of *F*,
- *F* is strictly convex if for every *x* in the domain of *F*,

2

- is positive definite

(Math Fundamentals cont'd.)

- As in the univariate case, if the negative of a multivariate function is (strictly) convex, then the function is (strictly) concave
- Note that properties of convexity are *global* in nature

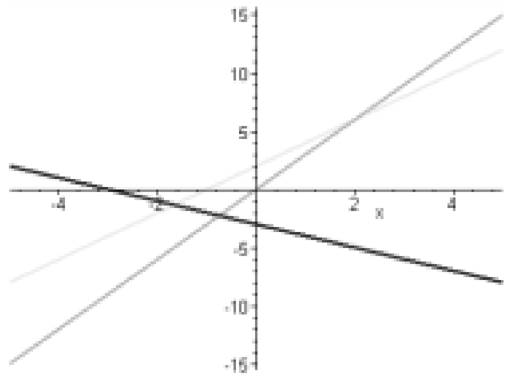
(Math Fundamentals cont'd.)

- Convex Sets and Functions — A Relation
  - We have convex and concave functions, but only convex sets. Are there concave sets? Strictly convex sets?
  - Let *g*() be a convex (possibly multivariate) *function*. Then the following *set* is a convex set
  - What about concave functions?

Univariate Functions (Tools in Our Bag of Tricks)

- Linear/Affine functions $f(x) = a + bx$
- where a and b are
- constants

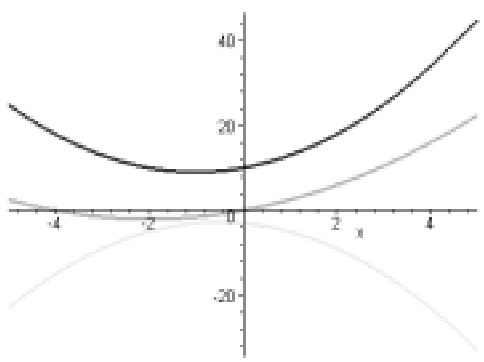
- 3x
- 2+2x
- -3-x



Univariate Functions (cont'd.)

- Quadratic functions
  $f(x) = a + bx + cx^2/2$
  where a, b, and c
  are constants.

- $2x+0.5x^2$
- $10+2x+x^2$
- $-3-x-x^2$

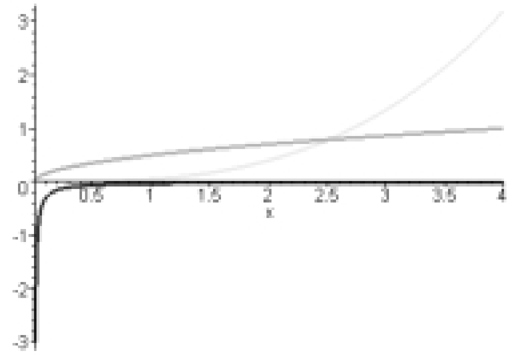


Univariate Functions (cont'd.)

- Constant elasticity functions

$$f(x) = \alpha x^{\varepsilon}$$

where α and ϵ are
- constants.



- $0.5x^{0.5}$
- $0.05x^{3}$
- $-0.03x^{-1}$

Univariate Functions (cont'd.)

This function gets its name because

$$\varepsilon = \frac{x}{\ } \frac{\partial f(x)}{\ } .$$

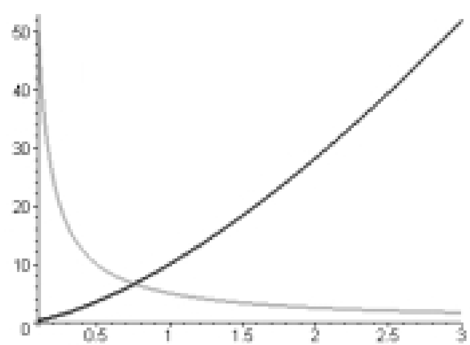
Univariate Functions (cont'd.)

- As ε approaches zero, the constant elasticity function approaches the natural logarithm of *x* if we interpret α as the appropriate function of *x*:

$$\ln(x) = \lim_{\varepsilon \to 0} \frac{x^{\varepsilon} - 1}{\varepsilon}$$

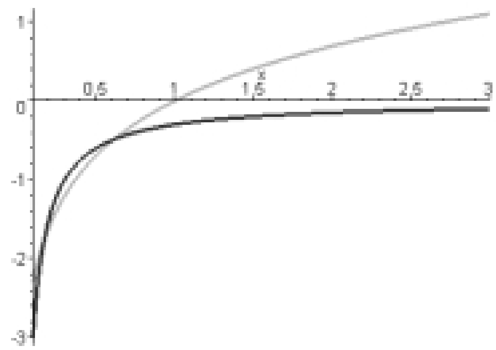
Univariate Functions (cont'd.)

- Common and limiting cases of the constant elasticity function



Univariate Functions (cont'd.)

- Less common and limiting cases of constant elasticity function



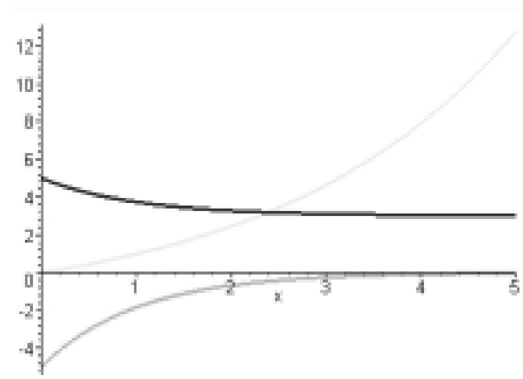
Univariate Functions (cont'd.)

- Exponential functions

where α, β, and γ are all constants. These functions are monotonic, and depending upon the sign of β, convex ($\geq$0) or concave ($\leq$0).

Univariate Functions (cont'd.)

- Exponential functions



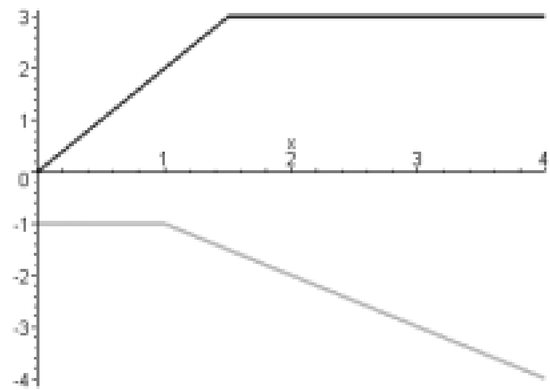
Univariate Functions (cont'd.)

- Linear response and plateau functions

where α and β are constants.

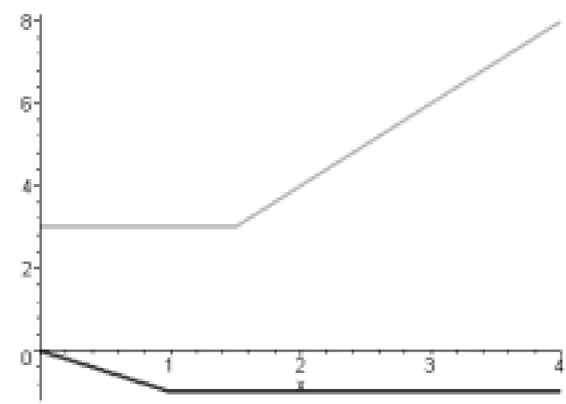
Univariate Functions (cont'd.)

- Linear response and plateau functions (min)



Univariate Functions (cont'd.)

- Linear response and plateau functions (max)



Multivariate Functions (More Tricks!)

- Multivariate linear functions

$$F(x) = \alpha_0 + \alpha' x = \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i$$

- Multivariate quadratic functions

$$F(x) = \alpha_0 + \alpha' x + \frac{1}{2} x' \beta x = \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \beta_{ij} x_i x_j$$

where $\beta_{ij} = \beta_{ji}$.

Multivariate Functions (Cont'd.)

- Cobb-Douglas functions

$$F(x) = \alpha \prod^{n} x_i^{\beta_i} = \alpha x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

where α and β_i are (strictly) positive constants. This function is increasing and concave so long as the sum of the exponents does not exceed 1.

Multivariate Functions (Cont'd.)

- Further, if the exponents sum to 1, then this function has constant returns to scale.

$$F(\lambda x) = \lambda F(x) \quad \forall \ \lambda \geq 0$$

- If they sum to less than one, then this function has decreasing returns to scale.

$$F(\lambda x) < \lambda F(x) \quad \forall \ \lambda > 1$$

- Note that the Cobb-Douglas is a multivariate generalization of the constant elasticity function.

Multivariate Functions (Cont'd.)

- Leontief functions

where the α_i are (usually positive) constants.

- Leontief functions are often employed via linear programming:

$$\text{maximize } y$$

Multivariate Functions (Cont'd.)

- Constant elasticity of substitution functions

$$F(x) = \alpha \left[\sum_{i=1}^{n} \beta_i x_i^{-\rho} \right]^{-1/\rho} = \alpha \left[\beta_1 x_1^{-\rho} + \beta_2 x_2^{-\rho} + \ldots + \beta_n x_n^{-\rho} \right]^{-1/\rho}$$

where α, β_i, and ρ are constants with α and β_i positive.

  - This function is increasing and (as written above) always has constant returns to scale.

Multivariate Functions (Cont'd.)

  - The pairwise elasticity of substitution between inputs is related to the exponent as follows:

$$\sigma = \frac{1}{\quad} \quad \text{or} \quad \rho = \frac{1-\sigma}{\quad}.$$

  - In the elastic case, $\sigma>1$, and in the inelastic case $0<\sigma<1$ ($\rho>0$, and $-1< \rho<0$, respectively)

Multivariate Functions (Cont'd.)

- As σ approaches 1, this function approaches the Cobb-Douglas
- When σ=0, the Leontief results

Multivariate Functions (Cont'd.)

- A decreasing returns to scale version of this function is obtained by introducing another positive constant:

$$F(x) = \alpha\left[\gamma + \sum^{n} \beta_i x_i^{-\rho}\right]^{-1/\rho} = \alpha\left[\gamma + \beta_1 x_1^{-\rho} + \cdots + \beta_n x_n^{-\rho}\right]^{-1/\rho}$$

- This is like having an additional input that is fixed. Do you see why?

$$F(x) = \alpha\left[\beta_0 x_0^{-\rho} + \sum^{n} \beta_i x_i^{-\rho}\right]^{-1/\rho}$$