Beyond Simple Optimization

- All models we have examined so far have been of the single-agent variety
  - This perspective is consistent with situations where there is a single decision-making entity (person, household, country, firm, or government)
  - It is also consistent with the case where the goals of all agents are consistent -- e.g., a national economy where the consumer wants the producers to be efficient

Beyond Simple Optimization (cont'd.)

  - When there is a single agent, then optimizing some objective makes conceptual sense
  - In the convex programming case, finding an optimum is equivalent to finding a KKT stationary point (i.e., the second-order conditions are satisfied automatically for a convex programming problem)

Beyond Simple Optimization (cont'd.)

- Consider the following problem:

$$\begin{aligned}&\text{minimize } F(x)\\&\text{subject to:}\\&0 \le x_i \qquad i = 1,\dots,n\end{aligned}$$

- The FOC's for this problem can be written as:

$$x_i \ge 0, \quad \frac{\partial F(x)}{\partial x_i} \ge 0 \text{ and } \left[\frac{\partial F(x)}{\partial x_i}\right] x_i = 0$$

Beyond Simple Optimization (cont'd.)

- The resulting problem is called a "complementarity problem"
- This is because both the gradient vector and the variable vector must be non-negative, and
  - When one is strictly positive, the other must be zero
  - So the set of strictly positive indices for *x* must be no greater than the *complement* of the set of strictly positive indices for the gradient

Beyond Simple Optimization (cont'd.)

- There are analogous complementarity problems for convex programs with more complicated constraint sets
- To see this it is easiest to start with
  - A bound constrained optimization problem, and then a
  - Generally constrained problem

Beyond Simple Optimization (cont'd.)

- Consider the following problem:

$$\begin{aligned} &\text{minimize } F(x) \\ &\text{subject to:} \end{aligned}$$

- The FOC's for this problem can be written as:

$$\left[\frac{\partial F(x)}{\partial x_i}\right]^{+}(x_i - l_i) - \left[\frac{\partial F(x)}{\partial x_i}\right]^{-}(u_i - x_i) = 0$$

Beyond Simple Optimization (cont'd.)

- Here the notation $[\ z\ ]^{+}$ indicates the maximum of z and zero (I.e., the "positive part of z")
- Likewise, $[\ z\ ]^{-}$ indicates the "negative part of z" and is the minimum of z and zero
- Thus, both $[\ z\ ]^{+}$ and $-[\ z\ ]^{-}$ are non-negative (note the minus sign in front of the negative part of z)
- Now consider the fully general problem

Beyond Simple Optimization (cont'd.)

- For the (convex programming) problem:

$$\text{minimize } F(x)$$
$$\text{subject to:}$$
$$v_j \geq C_j(x) \geq k_j \quad j = 1, \ldots, m$$

Beyond Simple Optimization (cont'd.)

- The first-order conditions can be written as:

$$\left[\frac{\partial F(x)}{\partial x_i} + \sum_{j=1}^{m}\left\{\lambda_j\left(\frac{\partial C_j(x)}{\partial x_i}\right) - \mu_j\left(\frac{\partial C_j(x)}{\partial x_i}\right)\right\}\right]^{+}(x_i - l_i) -$$

$$\left[\frac{\partial F(x)}{\partial x_i} + \sum_{j=1}^{m}\left\{\lambda_j\left(\frac{\partial C_j(x)}{\partial x_i}\right) - \mu_j\left(\frac{\partial C_j(x)}{\partial x_i}\right)\right\}\right]^{-}(u_i - x_i) = 0$$

$$C_j(x) \geq k_j,\ \lambda_j \geq 0,\ \lambda_j(C_j(x) - k_j) = 0,$$

$$v_j \geq C_j(x),\ \mu_j \geq 0,\ \mu_j(v_j - C_j(x)) = 0,$$

$$l_i \leq x_i \leq u_i$$

Beyond Simple Optimization (cont'd.)

- Once again, these complementarity conditions are nothing more fancy than the KKT first-order conditions set in a very general context
- We have just seen that *every* optimization problem can be written as a complementarity problem
- Now let us see what the general statement of a complementarity problem is

Beyond Simple Optimization (cont'd.)

- The Mixed Complementarity Problem (MCP to GAMS) is state as follows:

$$\text{Find } l \le x \le u \text{ so that:}$$
$$G_i(x)]^+(x_i - l_i) - [G_i(x)]^-(u_i - x_i) = 0$$

- which looks pretty simple, but isn't
- Notice that there are the same number of variables as equations

Beyond Simple Optimization (cont'd.)

- Also notice that if a variable is strictly between its bounds, then the *associated* equation must be zero
- Likewise, if an equation is strictly positive, then the associated variable must be on its lower bound, and
- If an equation is strictly negative, then the associated variable must be on its upper bound
- These sets cover all possibilities and again describe a complementary relationship between equations and variables

Beyond Simple Optimization (cont'd.)

- So, every optimization problem can be written as a complementarity problem through its FOC's
- Is every complementarity problem a system of FOC's for an optimization problem?
- No. E.g.,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} \infty \\ \infty \end{bmatrix}, \text{ and } G(x) = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

Beyond Simple Optimization (cont'd.)

- Q: Why am I sure this problem is not a set of FOC's for an optimization problem?
- A: If the problem is a set of FOC's for an optimization problem then the Jacobian of *G*(*x*) will either be
  - Symmetric (corresponding to the Hessian of some objective) or
  - Skew-symmetric (corresponding to the bordered Hessian for the constrained optimization problem)

Beyond Simple Optimization (cont'd.)

- Nonetheless, this problem is perfectly good, and has solution $x_1 = 4$, and $x_2 = 0$
- Bottom line – all optimization problems can be written as complementarity problems through their FOC's, but not all complementarity problems correspond to sets of FOC's for an optimization problem
- Q: So what!
- So there is some more general class of *economic* models that we consider using the complementarity formulation!

Multi-agent Models

- The type of model that we can create in the complementarity framework is the multi-agent model
- With a multi-agent model there are several agents that are simultaneously trying to optimize their goals
- There is some mechanism (a market in most economic models) that equilibrates the system so that their choices become consistent

Pure Trade – Competing Consumers

- What if we have a consumer?

$$\underset{x}{Maximize}\ \ u(x)$$
$$subject\ \ to:\ p'x \le I,\ x \ge 0.$$

- Recall the first-order conditions for this problem:

$$-\frac{\partial u(x)}{\partial x_i} + \lambda p_i \le 0,\ \ x_i \ge 0,\ \ x_i\left(\frac{\partial u(x)}{\partial x_i} - \lambda p_i\right) = 0,$$
$$-p'x + I \le 0,\ \ \lambda \ge 0,\ \ \lambda\left(p'x - I\right) = 0.$$

Competing Consumers (cont'd.)

- Notice again that these conditions can be written as a complementarity problem:

- What if we had a number of consumers all competing for the goods in an economy?
- What causes the consumer to adjust their demands so that markets clear?
- Where do they get their income?

Competing Consumers (cont'd.)

- Prices are imposed on consumers to make supply and demand equal, but there is another factor – income
- Prices also determine the income levels of consumers by valuing their initial holdings
- The idea is to write down consumers' first-order conditions combined with market clearing conditions as a complementarity problem

Pure Trade Example

- Let's say we have an economy consisting of
  - Two consumers -- Dick and Jane
  - Two goods -- bananas and coconuts
  - Dick owns 50 bananas and 10 coconuts
  - Jane owns 12 bananas and 40 coconuts

Pure Trade Example (cont'd.)

- The last thing we need to know is the preferences of our consumers which are Cobb-Douglas for this example with
  - Dick having a value share 0.4 for bananas
  - Jane having a value share 0.3 for bananas
- Wait a minute! What about their income?
- Their "income" or wealth in this case comes from their ownership of initial stocks of bananas and coconuts

Pure Trade Example (cont'd.)

- One last point – what is the unit of currency?
- This type of model where income and prices are both endogenous and all-encompassing cannot determine absolute price levels
- However, we can arbitrarily choose to value everything in the economy in terms of bananas
- Consider how to set this problem up in GAMS

Pure Trade Example (cont'd.)

- As usual, the first thing we will need is some sets:
  - SET good / banana,coconut /
  -         cons / dick,jane / ;
  - ALIAS (good,g) ;

- Then, we will need to specify the parameters of preference
  - TABLE alpha(good,cons)

	dick	jane
banana	0.4	0.3
coconut	0.6	0.7 ;

Pure Trade Example (cont'd.)

- Next, we specify the endowments that will define income
  - TABLE endow(good,cons)

	dick	jane
banana	50	12
coconut	10	40 ;

Pure Trade Example (cont'd.)

- Three sets of variables must be determined
  - POSITIVE VARIABLES
  - x(good,cons) Demand levels
  - lambda(cons) Marginal utility of wealth
  - p(good) Market prices ;
- Note that because preferences are Cobb-Douglas, demand must be strictly positive which we enforce with:
  - x.lo(good,cons) = 0.01 ;

Pure Trade Example (cont'd.)

- There are a corresponding three sets of equations
  - EQUATIONS
  - foc(good,cons) Utility max first-order conditions
  - budget(cons) Budget constraints
  - market(good) Market clearing conditions ;
- Structure of these equations is defined as usual:
  - foc(good,cons) .. –alpha(good,cons)*
  - prod(g,x(g,cons)**alpha(g,cons))/x(good,cons)
  - + lambda(cons)*p(good) =g= 0 ;

Pure Trade Example (cont'd.)

- Note that the individual agent problems are set up as *minimization* problems subject to greater than or equal to constraints
  - budget(cons) ..
  - –sum(good,p(good)*x(good,cons))
  - + sum(good,p(good)*endow(good,cons)) =g= 0 ;
- Finally, we have market clearing conditions:
  - market(good) .. sum(cons,endow(good,cons))
  - – sum(cons,x(good,cons)) =g= 0 ;

Pure Trade Example (cont'd.)

- Because absolute price levels cannot be determined in this type of model, a numeraire is set by forcing the price of bananas to be 1.0:
  - p.fx('banana') = 1 ;
- When we define the model, we must also define the correspondence between equations and variables:
  - model puretrade / foc.x,budget.lambda,market.p / ;
- Note that these pairs <eqn>.<var> must be compatibly indexed

Pure Trade Example (cont'd.)

- To solve the model, we use a new model type:
  - SOLVE puretrade USING MCP ;
- Note that there is no specification of a variable to be optimized nor a direction (i.e., minimize or maximize)
- That is because the problems do not have an objective (although in the current one we have two inconsistent ones!)

Pure Trade Example (cont'd.)

- The best solver for this class of problems (that is available to us) is PATH, and its output looks quite different from MINOS
- However, the format of printed solutions is essentially the same as with an optimization problem

Pure Trade Example (cont'd.)

- For instance, the equation output looks like
  - ---- EQU FOC Utility max first-order conditions

	LOWER	LEVEL	UPPER	MARGINAL
banana .dick	.	.	+INF	29.600
banana .jane	.	.	+INF	32.400
coconut.dick	.	.	+INF	18.500
coconut.jane	.	.	+INF	31.500

- Note that the marginals are now the *levels* for the complementary variables

Pure Trade Example (cont'd.)

- Similarly,
  - ---- EQU BUDGET Budget constraints

	LOWER	LEVEL	UPPER	MARGINAL
dick	.	.	+INF	0.302
jane	.	.	+INF	0.294

  - ---- EQU MARKET Market clearing conditions

	LOWER	LEVEL	UPPER	MARGINAL
banana	-62.000	-62.000	+INF	1.000
coconut	-50.000	-50.000	+INF	2.400

Pure Trade Example (cont'd.)

- The variables section is likewise similar:
  - ---- VAR X Demand levels

	LOWER	LEVEL	UPPER	MARGINAL
banana .dick	0.010	29.600	+INF	.
banana .jane	0.010	32.400	+INF	.
coconut.dick	0.010	18.500	+INF	.
coconut.jane	0.010	31.500	+INF	.

- Here, the marginals are the *levels* of the complementary equations

Pure Trade Example (cont'd.)

- Similarly,
  - ---- VAR P Market prices

	LOWER	LEVEL	UPPER	MARGINAL
banana	1.000	1.000	1.000	EPS
coconut	.	2.400	+INF	.

- ---- VAR LAMBDA Marginal utility of wealth

	LOWER	LEVEL	UPPER	MARGINAL
dick	.	0.302	+INF	.
jane	.	0.294	+INF	.

Pure Trade Example (cont'd.)

- So using the MCP framework, we can formulate models that include
  - Multiple agents with conflicting objectives
  - Additional equations (market clearing conditions in the example) that coordinate the actions of agents
- Notice that the variables associated with our market clearing conditions (prices) were *parameters* in the agents' behavioral problems