Random Variables in EU

- Specification of the random variables can universally be summarized by their joint probability distribution
- The form of the distributional information has a large impact on model formulation:
  - Continuous
  - Discrete
  - Mixed
  - Moments

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Random Variables (cont'd.)

- Continuous random variables are specified through a *density* function and a domain of integration
  - E.g., the joint normal density is

    $$-1$$

    where *C* is a constant, and its domain is all of *n*-dimensional space

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Random Variables (cont'd.)

- Discrete distributions are specified via a list of discrete points and probabilities $(\beta^k, p^k),\ k=1,...,K$
  - E.g., the binomial distribution is

    $$p^k = \binom{K}{k} \delta^k (1-\delta)^{K-k},\ k=0,1,...,K$$

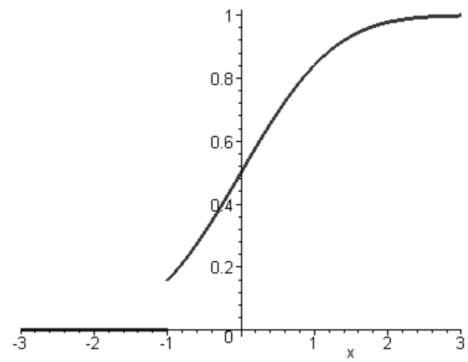
    where δ is a fixed value strictly between zero and one (Note that these probabilities do sum to 1.0!)
- If multivariate, these tend to be empirical distributions

Random Variables (cont'd.)

- Mixed distributions are specified as a mixture of a continuous and a discrete distribution
  - E.g., We could define a "truncated normal" distribution as having the univariate standard normal distribution for β>0, and having "point mass" at β=0 (What does the CDF look like?)
- These are quite uncommon as the fundamental variables
- Some payoff functions are similar (e.g., insurance)

Truncated Normal with Point Mass CDF

- CDF of a normal truncated from below with point mass at truncation point



Random Variables (cont'd.)

- A limited number of moments may be all that is available in some cases
  - E.g., perhaps only the mean vector and covariance are available
- Generally, more moments are better than fewer, but one must pay attention to the ultimate use in the math program (worry about convexity/concavity!)
- Gaussian quadrature and entropy may play a role

Random Variables (cont'd.)

- Gaussian quadrature and entropy – an aside
  - At times, one needs to have a discrete approximation to a distribution when only moments are available
    - Gaussian quadrature produces such an approximation that contains the fewest points (and least uncertainty)
    - Entropy produces such an approximation that contains the most points (and most uncertainty)

Random Variables (cont'd.)

- For either problem,
  - The first step is to generate a bunch of points in the domain for the random variables $\tilde{}^{k}$
  - The second step is to compute probabilities that satisfy some "moment constraints" that extremize some objective
- The difference between the two methods has to do with the choice of objective

Random Variables (cont'd.)

- With two random variables, moment constraints are:

$$\sum_{k=1}^{K} (\beta_1^k)^i (\beta_2^k)^j \, p^k = E\left[\beta_1^i \beta_2^j\right], \quad i, j = 0, \ldots, M;\; i + j \leq M$$

- In addition, we require that $p^k \geq 0$, and that these probabilities sum to 1.0
- For Gaussian quadrature, the objective is vacuous (there is none)
  - How many probabilities will be positive?

Random Variables (cont'd.)

- For maximum entropy, the objective is:

$$\text{maximize } -\sum^{K} p^k \ln(p^k)$$

- How many probabilities will be positive now?
  - Hint: look at the marginal entropy as some p^k approaches zero.

$$\lim -\frac{\partial p^k \ln(p^k)}{\partial p^k} = \lim \, [-1 - \ln(p^k)] = -\infty$$

Payoff Function

- The payoff function is entirely case specific and its construction employs a combination of
  - Accounting (to get the data)
  - Economics (to get appropriate behavioral implications of the data)

Implementations of Risky Choice Models

- A large number of alternative implementations of models of choice under risk appear in the literature
- The primary drivers of the choice of approach have historically been:
  - The form of the distributional information on the random variables
  - Expertise and software availability for the class of problem implied by the model

Risk Model Implementations

- The primary difficulty with this type of model is the objective function
  - Continuous case:

$$E[u(W(x,\beta))] = \int u(W(x,\beta))\, f(\beta) d\beta$$

    - Notice that this is a multi-dimensional integral
    - $f(\beta)$ denotes the *joint* distribution of the random variables

Implementation (cont'd.)

    - The integral will often have no closed form solution
    - Even if the integral does have a closed form solution, the resulting objective will be nonlinear
  - Most implementations of the above problem are based on :
    - Simplifications resulting from distributional choices
    - Approximations to the expected utility function
    - Approximations of the joint distribution

Implementation (cont'd.)

- Discrete case:

$$E[u(W(x,\beta))] = \sum_{i=1}^{n} u(W(x,\beta_i))\,p_i$$

  - In this case, the subscripts *i* do not indicate elements of the beta vector, but discrete realizations of the random variables (e.g., head or tail)
  - β_i denotes the vector of outcomes in "state of nature" *i*
  - p_i denotes the probability of state of nature *i*

Implementation (cont'd.)

- This formulation remains nonlinear, but no longer requires a closed form expression for a multi-fold integral
- This is the formulation that results if we make a discrete approximation to the joint distribution
- This is a handy formulation when you want to reflect data directly in the problem -- if the are observations from the real world, and $p_i = 1/n$, then we call this an empirical distribution

Implementation (cont'd.)

- For illustration, a variety of implementations of a particularly simple but theoretically important model of choice under risk -- the single-period portfolio model

$$\underset{x \geq 0}{\text{maximize}}\ E[u(W(x,\beta))] = E[u(\beta' x)]$$

$$\text{subject to}: \sum_{i=1}^{n} x_i \leq I$$

  - x_i denotes the amount of initial wealth, *I*, invested in asset *i*
  - The payoff function is bilinear in the random variables and the choice variables

Implementation (cont'd.)

- The Mean/Variance or E/V model

$$\underset{x \geq 0}{\text{maximize}}\ E[\beta]' x - \frac{\bar{\rho}}{2} x' E\{[\beta - E[\beta]][\beta - E[\beta]]'\} x$$

$$\text{subject to}: \sum^{n} x_i < I$$

  - Freund (1956) showed that if the beta's are distributed as joint normal random variables and if the utility function is a risk averse (negative) exponential then this problem produces identical results to the portfolio problem where rho is the coefficient of absolute risk aversion

Implementation (cont'd.)

- Meyer showed that if the beta's are joint normal, then there exists a rho that yields the optimum of the portfolio problem -- regardless of which risk averse utility function is the true one
- Notice the differences in these statements
  - Freund shows equivalence between E/V and the portfolio problem
  - Meyer shows that every optimum to a portfolio problem is optimal for the E/V problem with some value for rho

Implementation (cont'd.)

- Note that the above problem is a quadratic programming problem
- In Freund's time, these problems were *hard*
- Subsequent to Freund, a number of linear programming approximations to the expected utility (or E/V) problems were developed

Implementation (cont'd.)

- MOTAD (Hazell, 1971) or **M**ean **O**f **T**otal **A**bsolute **D**eviation -- One of the best known of the linear programming approximations to the E/V model
  - Assumptions
    - The payoff function is bi-linear in the random variables and choice variables (as in the portfolio problem)
    - The states of nature are equally likely (e.g., as with an empirical distribution)

Implementation (cont'd.)

- Motivation
  - Let $\bar{\beta}_i = \frac{1}{J}\sum^{J} \beta_{ij}$ denote the mean of random variable *i*
  - $E[W(x,\beta)] = \bar{\beta}'x = \sum^{n} \bar{\beta}_i x_i$ and
  - $ar(W(x,\beta)) = \frac{1}{J}\sum^{J}\left(\sum^{n} x_i(\beta_{ij} - \bar{\beta}_i)\right)^2$

Implementation (cont'd.)

- The idea with MOTAD is that we approximate the above variance with the following:

$$ar(W(x,\beta)) \approx \frac{1}{J}\sum_{j=1}^{J}\left|\sum_{i=1}^{n} x_i(\beta_{ij} - \bar{\beta}_i)\right|$$

- What part of the variance is being approximated with the absolute value?
- Why is this similar to a variance?
- How is this different from the variance?

Implementation (cont'd.)

- Notice that this measure of variance cannot be incorporated directly into the objective because it is not differentiable
- These non-differentiabilities can be avoided by introducing additional constraints (equalities and inequalities) and introducing additional variables

Implementation (cont'd.)

- To incorporate the absolute value based approximation to variance, we first introduce some variables and constraints

$$< some\ deviation > -d^{+} + d^{-} = 0$$

- We restrict the new variables to be non-negative

Implementation (cont'd.)

- Consider the optimum to this problem:

$$\underset{d^{+},d^{-}\geq 0}{\text{minimize}}\ d^{+} + d^{-}$$

subject to:

$+ \quad -$

Implementation (cont'd.)

- The MOTAD formulation:

$$\underset{x}{\text{minimize}} \sum_{j=1}^{J} (d_j^+ + d_j^-)$$

$$\text{subject to: } \sum_{i=1}^{n} x_i \bar{\beta}_i \geq \mu$$

$$\sum_{i=1}^{n} x_i (\beta_{ij} - \bar{\beta}_i) - d_j^+ + d_j^- = 0$$

$$\sum_{i=1}^{n} x_i \leq I, \; x_i \geq 0, \, d_j^+ \geq 0, \, d_j^- \geq 0.$$

Implementation (cont'd.)

- Notice that we are now minimizing the approximation to variance subject to "doing well enough" on the mean
- Does this problem produce the same optimal x_i values as if we were minimizing the MOTAD measure of variance?
- Where is the risk aversion level?
- How does this relate to the E/V problem?

- Problem -- this approach penalizes deviations above and below the mean equally

Implementation (cont'd.)

- Target MOTAD (Tauer, 1983)
  - Similar to MOTAD, but does not penalize deviations above "target"
  - Requires user to specify a target as well as a mean level of income

Implementation (cont'd.)

- Formulation of Target MOTAD:

$$\underset{x}{\text{minimize}} \sum_{j=1}^{J} d_j^-$$

$$\text{subject to: } \sum_{i=1}^{n} x_i \beta_i \geq \mu$$

$$\sum_{i=1}^{n} x_i \beta_{ij} - T + d_j^- \geq 0$$

$$\sum_{i=1}^{n} x_i \leq I, \; x_i \geq 0, \; d_j^- \geq 0.$$

Implementation (cont'd.)

- Other linear programming based models of choice under risk
    - MaxiMin Criterion
    - Safety-first models
    - Etc.
  - These tend to focus on extremes -- i.e., what happens in the worst case

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Implementation (cont'd.)

- Mean/Variance Models -- Two Formulations

$$\underset{x}{\text{maximize}}\ \frac{1}{J}\sum_{j=1}^{J} W_j - \frac{\rho}{2J}\sum_{j=1}^{J}\left(W_j - \frac{1}{J}\sum_{k=1}^{J} W_k\right)^2$$

$$\text{subject to}: \sum_{i=1}^{n} x_i \beta_{ij} = W_j$$

$$\sum^{n} x_i \le I,\ \ x_i \ge 0.$$

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Implementation (cont'd.)

- Formulation 2

$$\underset{x}{\text{maximize}} \ \frac{1}{J}\sum_{j=1}^{J}\sum_{i=1}^{n} x_i \beta_{ij} - \frac{\rho}{2J}\sum_{j=1}^{J}\left(\sum_{h=1}^{n}\sum_{i=1}^{n} x_i(\beta_{ij}-\overline{\beta}_i)(\beta_{hj}-\overline{\beta}_h)x_h\right)$$

$$\text{subject to:} \ \sum_{i=1}^{n} x_i \le I, \ x_i \ge 0,$$

Implementation (cont'd.)

or in matrix notation

$$\underset{x}{\text{maximize}} \ \overline{\beta}'x - \frac{\overline{\rho}}{2}x'E[(\beta-\overline{\beta})(\beta-\overline{\beta})']x$$

Implementation (cont'd.)

- Notice that we don't need to know the joint distribution of the betas, only the mean and covariance matrix
- Thus, this approach can be applied in the continuous or discrete cases, or even when only the moments (not the distribution) are known
- All of the E/V problems above are (strictly convex) quadratic programming problems for risk averse agents

Implementation (cont'd.)

- Direct Expected Utility Models (Lambert and McCarl, 1985)
  - Based on directly implementing the expected utility model
  - Joint distribution of random variables must either be discrete or approximate as discrete

Implementation (cont'd.)

- Formulation of portfolio model:

$$\underset{x,W}{\text{maximize}} \sum_{j=1}^{J} p_j u(W_j)$$

$$\text{subject to}: \sum_{i=1}^{n} x_i \beta_{ij} = W_j$$

$$\sum_{i=1}^{n} x_i \le I,\ x_i \ge 0.$$

Implementation (cont'd.)

- Or more generally:

$$\underset{x,W}{\text{maximize}} \sum_{j=1}^{J} p_j u[W(x, \beta_j)]$$

$$\text{subject to}: \sum^{n} x_i \le I,\ x_i \ge 0.$$

- Generalization of each of these problems to the case of more general constraints (than the portfolio problem) is straightforward