Nonparametric Efficiency Testing (cont'd.)

- More often than not, there will be a mixture of fixed and variable inputs -- then we solve the following:

$$\underset{x_1,\ldots,x_t,\lambda^1,\ldots,\lambda^K}{\text{minimize}} \sum_{n=1}^{t} w_n^0 x_n$$

$$\text{subject to}: \sum_{k=1}^{K} u^k \lambda^k \geq u^0$$

$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n \quad \mathit{for}\ 1 \leq n \leq t$$

$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n \quad \mathit{for}\ n > t, \quad \sum_{k=1}^{K} \lambda^k = 1,\ \lambda^k \geq 0$$

Nonparametric Efficiency Testing (cont'd.)

- If we apply this test for firm 1 where input 1 is treated as fixed, we get:

$$\text{minimize } 2x_2$$

$$7\lambda^1 + 7\lambda^2 + 14\lambda^3 + 14\lambda^4 \geq 7$$

$$4\lambda^1 + 1\lambda^2 + 8\lambda^3 + 8\lambda^4 \leq 4$$

$$3\lambda^1 + 5\lambda^2 + 5\lambda^3 + 6\lambda^4 \leq x_2$$

$$-^{1} \quad -^{2} \quad -^{3} \quad -4^{7} \quad -^{k}$$

Nonparametric Efficiency Testing (cont'd.)

- The optimum for this problem has an objective value of 6
- Does firm 1 appear to be minimizing variable costs? (Remember that firm 1 was using 3 units of input 2)
- Recall that when both inputs were variable firm 1 appeared to be inefficient

Nonparametric Efficiency Testing (cont'd.)

- Thus far, we have seen tests for
  - Technical efficiency in the production of one output
  - Allocative efficiency in the minimization of costs
  - There are three other types of efficiency tests:
    - Multiple output technical efficiency
    - Profit maximization
    - Revenue maximization, given input levels

Nonparametric Efficiency Testing (cont'd.)

- Multiple output technical efficiency test:

$$\underset{\tau,\lambda^1,\ldots,\lambda^K}{\text{maximize}} \quad \tau$$

$$\text{subject to}: \sum_{k=1}^{K} u_m^k \lambda^k \geq u_m^0 \tau$$

$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n^0$$

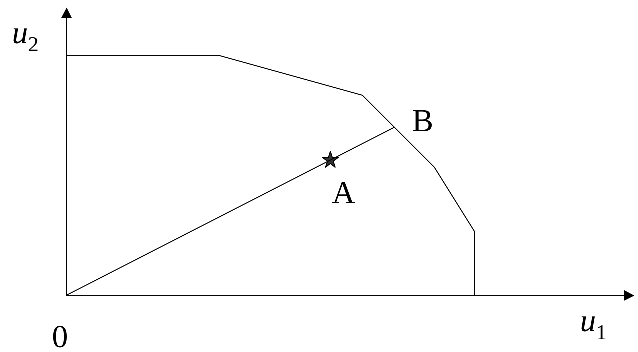
$$\sum_{k=1}^{K} \lambda^k = 1,\ \lambda^k \geq 0$$

Nonparametric Efficiency Testing (cont'd.)

- Here, we are asking if the entire vector of outputs can be scaled up by a single factor
- We are not asking whether more of any output could be produced
- What will the optimal objective value be for
  - An efficient producer?
  - An inefficient producer?

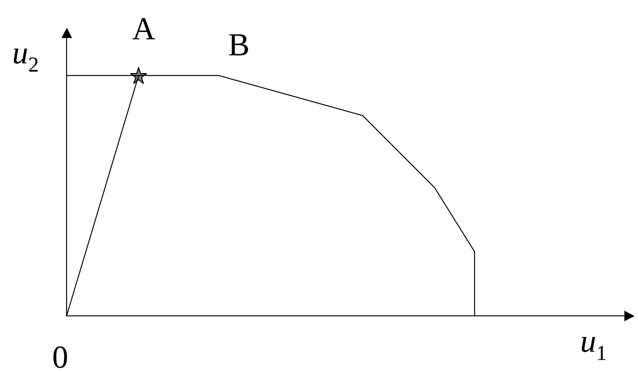
Nonparametric Efficiency Testing (cont'd.)

- Note that we are testing each output at a time for efficiency (0B/0A) in this case



Nonparametric Efficiency Testing (cont'd.)

- Here is a case where this test produces an unintuitive result



Nonparametric Efficiency Testing (cont'd.)

- For each *m*' we could solve the following:

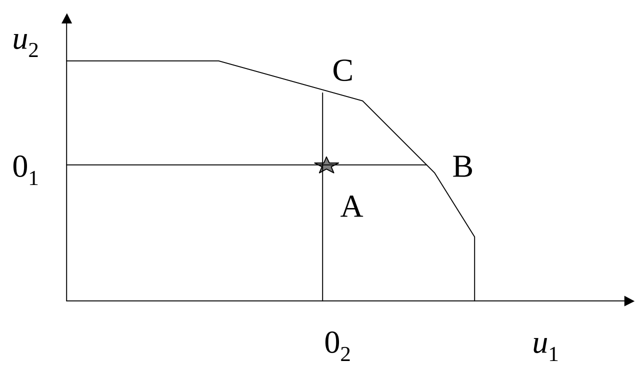
$$\underset{\tau_{m'},\lambda^1,\ldots,\lambda^K}{\text{maximize}} \;\; \tau_{m'}$$

$$\text{subject to}: \sum_{k=1}^{K} u_m{}^k \lambda^k \geq u_{m'}^0 \tau_{m'}$$

$$\sum_{k=1}^{K} u_m{}^k \lambda^k \geq u_m^0$$

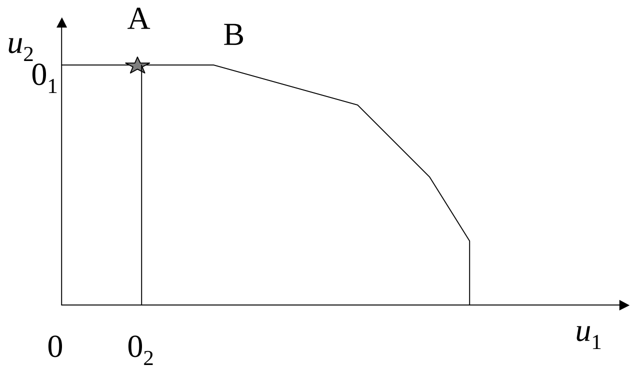
$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n^0, \;\; \sum_{k=1}^{K} \lambda^k = 1, \; \lambda^k \geq 0$$

Nonparametric Efficiency Testing (cont'd.)



Nonparametric Efficiency Testing (cont'd.)

- Now consider our unintuitive example



Nonparametric Efficiency Testing (cont'd.)

- Profit maximization -- looks much like cost min test

$$\underset{x_1,\ldots,x_N,u_1,\ldots,u_M,\lambda^1,\ldots,\lambda^K}{\text{maximize}} \sum_{m=1}^{M} p_m^0 u_m - \sum_{n=1}^{N} w_n^0 x_n$$

$$\text{subject to}: \sum_{k=1}^{K} u_m^k \lambda^k \geq u_m$$

$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n, \quad \sum_{k=1}^{K} \lambda^k = 1, \; \lambda^k \geq 0$$

Nonparametric Efficiency Testing (cont'd.)

- Notice that this is the multiple output case
- Notice the similarity with the cost min test
- How would you identify inefficiency?
- Where is the data in the problem that is relevant to the individual firm?
- How does this change with fixed inputs?

Nonparametric Efficiency Testing (cont'd.)

- If we apply this test for firm 2 where the output price:

$$\text{minimize } 4u - 4x_1 - 3x_2$$

$$7\lambda^1 + 7\lambda^2 + 14\lambda^3 + 14\lambda^4 \geq u$$

$$4\lambda^1 + 1\lambda^2 + 8\lambda^3 + 8\lambda^4 \leq x_1$$

$$3\lambda^1 + 5\lambda^2 + 5\lambda^3 + 6\lambda^4 \leq x_2$$

$$^{1} \quad ^{2} \quad ^{3} \quad ^{4} \quad ^{k}$$

Nonparametric Efficiency Testing (cont'd.)

- The optimal solution for this problem has an objective value of 9
- Is firm 2 efficient?
- Yes – observed profit was 9
- How would this change if input 1 was fixed?

Nonparametric Efficiency Testing (cont'd.)

- Test of profit max with input 1 fixed – Firm 2

$$\text{minimize } 4u - 2x_2$$

$$7\lambda^1 + 7\lambda^2 + 14\lambda^3 + 14\lambda^4 \geq u$$

$$4\lambda^1 + 1\lambda^2 + 8\lambda^3 + 8\lambda^4 \leq 1$$

$$3\lambda^1 + 5\lambda^2 + 5\lambda^3 + 6\lambda^4 \leq x_2$$

$$\lambda^1 \quad \lambda^2 \quad \lambda^3 \quad \lambda^4 \quad \lambda^k$$

- New optimal objective is 18, and firm 2 is efficient

Nonparametric Efficiency Testing (cont'd.)

- Revenue maximization test

$$\underset{u_1,\ldots,u_M,\lambda^1,\ldots,\lambda^K}{\text{maximize}} \sum_{m=1}^{M} p_m^0 u_m$$

$$\text{subject to}: \sum_{k=1}^{K} u_m{}^k \lambda^k \geq u_m$$

$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n^0, \quad \sum_{k=1}^{K} \lambda^k = 1,\ \lambda^k \geq 0$$

Nonparametric Efficiency Testing (cont'd.)

- Again, this is similar to cost min/profit max problems and looks as follows for firm 2:

$$\text{minimize } 4u$$

$$7\lambda^1 + 7\lambda^2 + 14\lambda^3 + 14\lambda^4 \geq u$$

$$4\lambda^1 + 1\lambda^2 + 8\lambda^3 + 8\lambda^4 \leq 1$$

$$3\lambda^1 + 5\lambda^2 + 5\lambda^3 + 6\lambda^4 \leq 5$$

Nonparametric Efficiency Testing (cont'd.)

- In the single output case, is the revenue test strange?
- Note that technical inefficiency implies profit and revenue maximization inefficiency
- Cost minimization inefficiency implies profit maximization inefficiency
- Revenue maximization inefficiency implies profit maximization inefficiency

Nonparametric Efficiency Testing (cont'd.)

- How do we add the assumption of constant returns to scale?
- There is a problem with the profit maximization test when all inputs are variable and we assume constant returns to scale. What is it?

Nonparametric Efficiency Testing (cont'd.)

- Want to learn more? Then read:
  - Farrell, M.J., "The Measurement of Productive Efficiency," *Journal of the Royal Statistical Society* 120(3):253-290
  - Varian, H.R., 1984. "The Nonparametric Approach to Production Analysis," *Econometrica*, 52(3):579-597
  - Färe, R., S. Grosskopf, and H. Lee, 1990. "A Nonparametric Approach to Expenditure-constrained Profit Maximization, *American Journal of Agricultural Economics*, 72:574-581

Producer Problems -- Complications

- Incorporating additional activities/constraints in producer problems -- a question of scope
  - What makes producer problems interesting is their interactions with the world
    - Market interactions -- purchases and sales of inputs or outputs
    - Resource transactions -- factor rental, service hiring, borrowing

Producer Problems -- Complications

- Regulations
  - Prohibitions or limits on use of some inputs
  - Prohibitions or limits on the production of some outputs
- Contracts with suppliers of input, or demanders of outputs
- Formulation of constraints and objective must reflect
  - Producer incentives
  - Tradeoffs available to producers

Endogenous Price Models

- In some cases, price taking behavior is not the appropriate perspective, e.g.,
  - Where producers have market power
  - Models of entire sectors wherein we know there is a relationship between supply and demand quantities and market prices

Endogenous Price Models (cont'd.)

- The key is to know what the relationship between price and quantity is
  - Demand, Q(*P*), is typically known, but
  - What we need is inverse demand, *P*(*Q*); so, the relationship must be invertible
    - E.g., $Q(P) = \alpha P^{\varepsilon}$ and $P(Q) = (Q/\alpha)^{1/\varepsilon}$

Endogenous Price Models (cont'd.)

- In optimization modeling, endogenous price models are often used to model partial equilibrium problems
- Simple example:
  - Let's say we know that demand is $D(P) = aP^b$
  - and supply is $S(P) = cP^d$

Endogenous Price Models (cont'd.)

- Now consider the problem:

$$\underset{S,D}{\text{maximize}} \quad \frac{b}{1+b} D^{(1+b)/b} a^{-1/b} - \frac{d}{1+d} S^{(1+d)/d} c^{-1/d}$$

$$\text{subject to: } D - S \leq 0$$

- The first-order conditions for this problem are:

  - $(D/a)^{1/b} - P = 0$ (where P is the shadow price on the constraint),
  - $(S/c)^{1/d} - P = 0$, and
  - $D - S \leq 0$

Endogenous Price Models (cont'd.)

- Notice that we can use algebra to re-express the first order condition for D as $D = aP^b$
- And likewise for S, $S = cP^d$
- So, the shadow price on the constraint is acting like the market price and the relationships between Supply, Demand, and Market Price have been modeled

Endogenous Price Models (cont'd.)

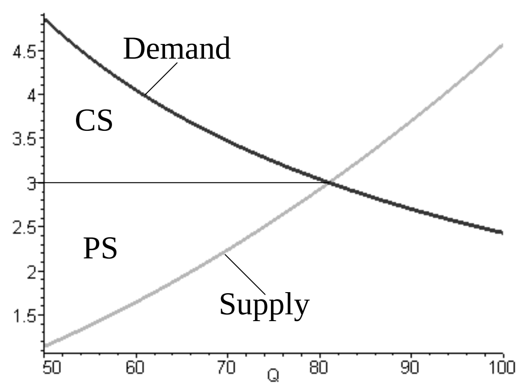
- How did we do this?

$$\underset{S,D}{\text{maximize}} \quad \int_0^D (d/a)^{1/b}\, dd - \int_0^S (s/c)^{1/d}\, ds$$
$$\text{subject to: } D - S \le 0$$

- So, our objective is the "integral under the inverse demand curve" less the "integral under the inverse supply curve" – producer's plus consumer's surplus

Endogenous Price Models (cont'd.)

- Graphically, here is what is going on



Endogenous Price Models (cont'd.)

- It is straightforward to adapt this model to analysis of *spatial* equilibrium problems
  - The base is a transportation problem, transhipment problem, or transhipment problem with processing
  - Demand and (or) supply are no longer treated as fixed, but rather integrals of the inverse relationships are incorporated in the objective

Endogenous Price Models (cont'd.)

- For example, the transportation model with variable supplies and demands becomes:

$$\underset{S_i, D_j}{\text{maximize}} \sum_{j=1}^{m} \int_0^{D_j} P_j(d_j)\,dd_j - \sum_{i=1}^{n} \int_0^{S_i} P_i(s_i)\,ds_i - \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to: } \sum_{i=1}^{n} x_{ij} \le D_j,\ S_i - \sum_{j=1}^{m} x_{ij} \le 0,\ x_{ij} \ge 0.$$

Endogenous Price Models (cont'd.)

- This formulation has the same interpretation as before -- producers' plus consumers' surplus
  - There is an adjustment for inter-industry costs
  - Graphical illustration is no longer feasible
- This type of model is frequently used to study markets for a single good or a sector (partial equilibrium)