Nonparametric Efficiency Testing

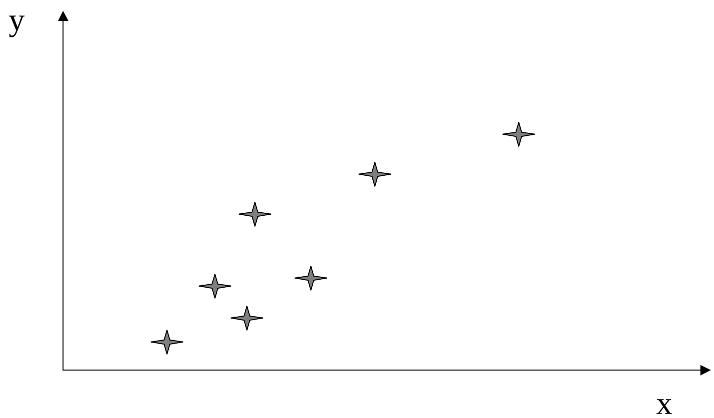
- A linear programming based approach to efficiency assessment
  - Cross section or time series input/output data viewed as observations from the technically feasible set
    - Assumptions for what follows includes that each observation comes from the same technology set
    - Modifications exist to account for technical progress

Nonparametric Efficiency Testing (cont'd.)

- The easiest way to think of efficiency is by looking at a single input/single output system
- Which of the following observations are efficient?

Observation	Output	Input
1	1	3
2	3	5
3	2	6
4	6	6
5	3	9
6	7	11
7	8	15

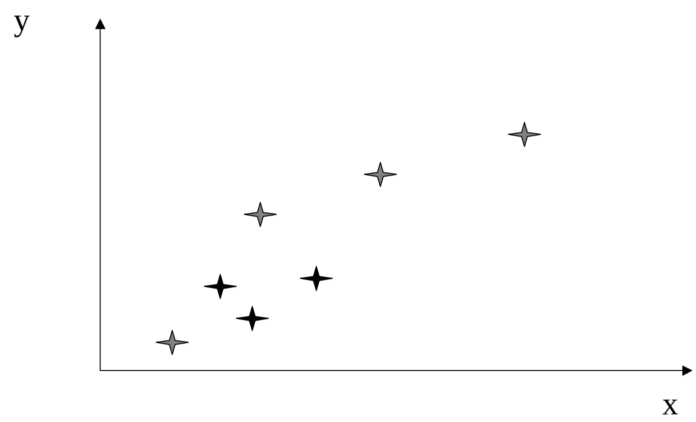
Nonparametric Efficiency Testing (cont'd.)



Nonparametric Efficiency Testing (cont'd.)

- What can you say about the efficiency of these observations?
- Are some observations more efficient than others?

Nonparametric Efficiency Testing (cont'd.)



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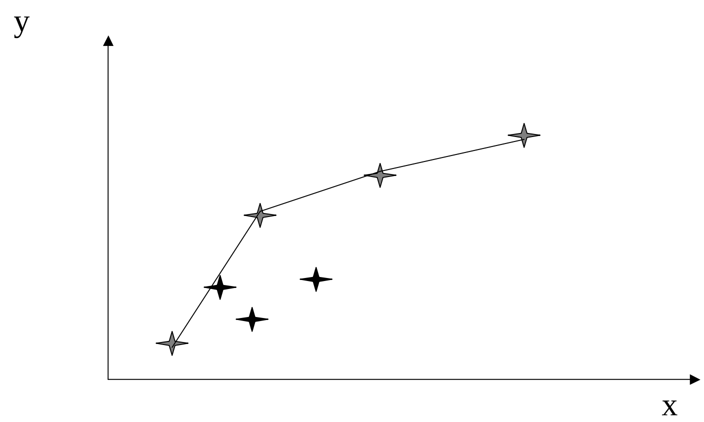
Nonparametric Efficiency Testing (cont'd.)



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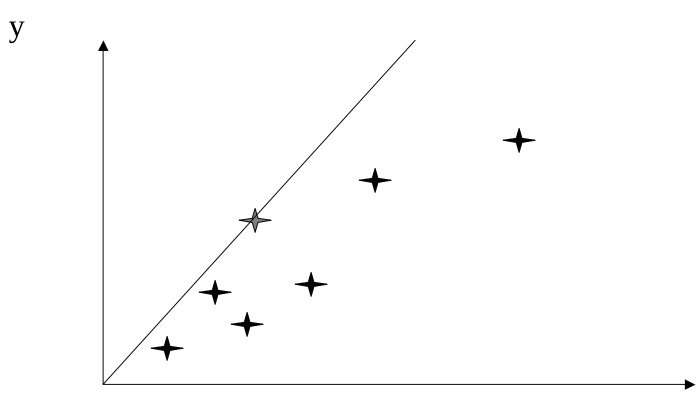
Nonparametric Efficiency Testing (cont'd.)

- What are you assuming about returns to scale?
- What happens if you change your returns to scale assumption?

Nonparametric Efficiency Testing (cont'd.)



Nonparametric Efficiency Testing (cont'd.)

- What else are you assuming?
  - Are you assuming anything about prices?
  - Can you generalize these ideas to higher dimensions (i.e., multiple inputs and/or multiple outputs)?

Nonparametric Efficiency Testing (cont'd.)

- Consider a two input/one output case
- Which of the following observations are efficient?

Observation	Output	Input 1	Input 2
1	5	1	1
2	5	2	1
3	6	1	2
4	8	1	3
5	8	2	2
6	9	3	3
7	10	4	2

Nonparametric Efficiency Testing (cont'd.)

- Which of the following observations are efficient?

Observation	Output	Input 1	Input 2
1	5	1	1
2	5	2	1

- Note that #2 uses more of input 1, the same amount of input 2, and produces no more output than #1. So, #2 is inefficient.

Nonparametric Efficiency Testing (cont'd.)

- Which of the following observations are efficient?

Observation	Output	Input 1	Input 2
1	5	1	1
3	6	1	2
4	8	1	3

- Note that input use for observation #3 is the average of input use for observations #1 and #4. However, the average output level for observations #1 and #4 is 6.5; strictly greater than observed output for #3. Hence we conclude #3 appears to be inefficient.

Nonparametric Efficiency Testing (cont'd.)

- Which of the following observations are efficient?

Observation	Output	Input 1	Input 2
1	5	1	1
5	8	2	2
6	9	3	3

- Note that the input usage for #5 is the average of the input usage for #1 and #6, but the output production (8) is greater than the average output 7 = (5+9)/2.
- What are we assuming about returns to scale?

Nonparametric Efficiency Testing (cont'd.)

- If we assume constant returns to scale (CRTS), what happens?

Observation	Output	Input 1	Input 2
1	5	1	1
5	8	2	2
6	9	3	3

- With CRTS, we can double inputs and outputs for observation 1. So, with 2 units of each input we can get 10 units of output. Similarly, 3 units of each input we get 15 units of output. So, #5 and #6 are inefficient.

Nonparametric Efficiency Testing (cont'd.)

- Let us formalize our assumptions:
  - Free disposability of inputs and outputs
  - Convexity of input requirements and production possibilities
  - No errors in the data

Nonparametric Efficiency Testing (cont'd.)

- Assume that you have a bunch of data for K different firms indexed by $k=1,\dots,K$
- The output of firm k is denoted by u^k and inputs are denoted by x_n^k where n indexes inputs
- Let's not assume constant returns to scale
- Consider the set of technically feasible input combinations that produce a given amount of output u^0

Nonparametric Efficiency Testing (cont'd.)

- Associate a "weight" with each observation (the *single* weight applies to all inputs and outputs)
- The technically feasible set of inputs producing u^0 is:

$$(u^0) = \left\{ x = (x_1,\ldots,x_N) \mid \sum_{k=1}^{K} u^k \lambda^k \geq u^0 \right.$$
$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n \qquad n = 1,\ldots,N$$
$$\left. \sum_{k=1}^{K} \lambda^k = 1,\ \lambda^k \geq 0\ k = 1,\ldots,K \right\}$$

Nonparametric Efficiency Testing (cont'd.)

- The role of the λ^k's is as weights in forming a convex combination of the input/output vectors (The final line makes the weights non-negative and sum to unity)
- The first inequality defining the set ensures that
  - At least u^0 will be produced by the linear combination of observed input/output vectors
  - Use of the inequality implies free disposal of output

Nonparametric Efficiency Testing (cont'd.)

- The second set of inequalities:
  - Ensure that at least as much input as the convex combination is used
  - Embody free input disposal (because it is an inequality)

Nonparametric Efficiency Testing (cont'd.)

- For the data listed above, this set is:

$$
\begin{aligned}
(u^0) = \{(x_1, x_2) \mid \; & 5\lambda^1 + 5\lambda^2 + 6\lambda^3 + 8\lambda^4 + 8\lambda^5 + 9\lambda^6 + 10\lambda^7 \geq u^0 \\
& 1\lambda^1 + 2\lambda^2 + 1\lambda^3 + 1\lambda^4 + 2\lambda^5 + 3\lambda^6 + 4\lambda^7 \geq x_1 \\
& 1\lambda^1 + 1\lambda^2 + 2\lambda^3 + 3\lambda^4 + 2\lambda^5 + 3\lambda^6 + 2\lambda^7 \geq x_2
\end{aligned}
$$

Nonparametric Efficiency Testing (cont'd.)

- Note that this set is a function of the output level u^0
- Efficiency has something to do with being on the *boundary* of this set
- The technical efficiency test asks the question, given the input vector, could higher output have been produced?

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Nonparametric Efficiency Testing (cont'd.)

- To answer this question, we could solve the following problem:

$$\underset{u,\lambda^1,\ldots,\lambda^K}{\text{maximize}}\ u$$

$$\textit{subject to}: \sum_{k=1}^{K} u^k \lambda^k \geq u$$

$$\sum_{k=1}^{K} x_i^k \lambda^k \leq x_i^0$$

$$\sum_{k=1}^{K} \lambda^k = 1,\ \lambda^k \geq 0$$

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Nonparametric Efficiency Testing (cont'd.)

- Note that x_i^0 are the fixed input levels for "firm 0" whose efficiency is being tested and whose data is included in k=1,…,K
- Can you argue that there is a lower bound for the optimal u? What is that bound?
- How should we decide if "firm 0" is efficient?
- What are we assuming about returns to scale?

Nonparametric Efficiency Testing (cont'd.)

- If we apply this test to the two input data and test for efficiency of firm 2, we solve the following problem

maximize u

$$5\lambda^1 + 5\lambda^2 + 6\lambda^3 + 8\lambda^4 + 8\lambda^5 + 9\lambda^6 + 10\lambda^7 \geq u$$
$$1\lambda^1 + 2\lambda^2 + 1\lambda^3 + 1\lambda^4 + 2\lambda^5 + 3\lambda^6 + 4\lambda^7 \leq 2$$
$$1\lambda^1 + 1\lambda^2 + 2\lambda^3 + 3\lambda^4 + 2\lambda^5 + 3\lambda^6 + 2\lambda^7 \leq 1$$

Nonparametric Efficiency Testing (cont'd.)

- The optimal solution for this problem is *u*=5
- What do we conclude?
- Does this agree with your intuition?
- Note that this was a test of technical efficiency
- Under what could this really be optimal?

Nonparametric Efficiency Testing (cont'd.)

- What are you assuming about prices?
  - Technical efficiency
  - Allocative efficiency

Nonparametric Efficiency Testing (cont'd.)

- Consider the following price and quantity data:
  - Output Input 1 Input 2
  - (p,q) (p,q) (p,q)
  - 5, 7 3, 4 2, 3
  - 4, 7 4, 1 3, 5
  - 4, 14 2, 8 3, 5
  - 3, 14 2, 8 1, 6

Nonparametric Efficiency Testing (cont'd.)

- Output Input 1 Input 2
- (q) (p,q) (p,q)
- 7 3, 4 2, 3
- 7 4, 1 3, 5
- Note that for the first and second observations the input mix is different, but output is the same.
- The **cost** of inputs for firm 1 is 3x4+2x3=18.
- The cost if firm 1 used firm 2's inputs=3x1+2x5=13.
- Hence, firm 1 is inefficient.
- Firm 2: observed cost is 4x1+3x5=19, achievable= 4x4+3x3=25. So firm 2, appears to be efficient.

Nonparametric Efficiency Testing (cont'd.)

Output	Input 1	Input 2
(q)	(p,q)	(p,q)
14	2, 8	3, 5
14	2, 8	1, 6

- Now compare firms 3 and 4. Note that firm 4 is inefficient in the way that made us uncomfortable when we looked closely at technical efficiency (more input, same output).
- Firm 4: observed costs=2x8+1x6=22, achievable costs=2x8+1x5=21.

Nonparametric Efficiency Testing (cont'd.)

- Clearly, we need a test that takes into account the *value* of inputs
- Now our question becomes how low could we drive costs and still produce as much as "firm 0"?
- Input levels now become variables in our testing problem, and the output level is fixed
- When testing firm 0, we always use firm 0 prices in our measurement of costs

Nonparametric Efficiency Testing (cont'd.)

- To test for cost minimization efficiency, we solve:

$$\underset{x_1,\ldots,x_N,\lambda^1,\ldots,\lambda^K}{\text{minimize}} \sum_{n=1}^{N} w_n^0 x_n$$

$$\text{subject to}: \sum_{k=1}^{K} u^k \lambda^k \geq u^0$$

$$\sum_{k=1}^{K} x_n^k \lambda^k \leq x_n$$

$$\sum_{k=1}^{K} \lambda^k = 1,\ \lambda^k \geq 0$$

Nonparametric Efficiency Testing (cont'd.)

- If we apply this test to the two input data and test for efficiency of firm 1, we solve the following problem

$$\text{minimize } 3x_1 + 2x_2$$

$$7\lambda^1 + 7\lambda^2 + 14\lambda^3 + 14\lambda^4 \geq 7$$

$$4\lambda^1 + 1\lambda^2 + 8\lambda^3 + 8\lambda^4 \leq x_1$$

$$3\lambda^1 + 5\lambda^2 + 5\lambda^3 + 6\lambda^4 \leq x_2$$

$$^{1} \quad ^{2} \quad ^{3} \quad ^{4} \quad ^{k}$$

Nonparametric Efficiency Testing (cont'd.)

- The optimal solution for this problem has an objective value of 13, with inputs at levels 1 and 5
- Does it appear that firm 1 is efficient?
- What should the objective be if firm 1 is efficient?
- Can we attribute some "degree" of inefficiency? (How about noting 100x18/13=138 and saying that costs for firm 1 are 38% too high?)

Nonparametric Efficiency Testing (cont'd.)

- If we apply this test for efficiency to firm 2, we solve the following problem

$$\text{minimize } 4x_1 + 3x_2$$

$$7\lambda^1 + 7\lambda^2 + 14\lambda^3 + 14\lambda^4 \geq 7$$

$$4\lambda^1 + 1\lambda^2 + 8\lambda^3 + 8\lambda^4 \leq x_1$$

$$3\lambda^1 + 5\lambda^2 + 5\lambda^3 + 6\lambda^4 \leq x_2$$

$$^{1} \quad ^{2} \quad ^{3} \quad ^{4\,7} \quad ^{k}$$

Nonparametric Efficiency Testing (cont'd.)

- The optimal solution for this problem has an objective value of 19, with inputs at levels 1 and 5
- Does it appear that firm 2 is efficient?
- What should the objective be if firm 2 is efficient?
- Will we ever find "hyper-efficiency" (i.e., minimum feasible costs higher than observed)?